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NEW METHOD IN THE THEORY OF RELAXATION PROCESSES FOR NON-HOMOGENOUS MEDIA

The Chapman–Enskog method for solving kinetic equations is generalized for the case of the presence of relaxation processes, which can take place in a spatially homogenous system. The investigation is based on the Bogolyubov idea of the functional hypothesis. For the first time, the relaxation processes are investigated at their final stage by going beyond a linearization and introducing a new small parameter in the theory. The proposed theory generalizes the known Grad approach, the main lack of which is the absence of a small expansion parameter. The conditions for the local equilibrium approximation validity and the role of the additional small parameters in the approximate solution of integral equations are investigated. As applications, the following problems are investigated: the formation of dissipative hydrodynamic fluxes in a simple gas (the Maxwell relaxation), the two-fluid hydrodynamics of completely ionized plasma (the relaxation of component temperatures and velocities), hydrodynamics of a system of polarons in polar dielectrics and semiconductors (temperature and velocity relaxation of the electron system), and hydrodynamics of a phonon system in dielectrics (phonon drift velocity attenuation due to umklapp processes).

Keywords: generalization of the Chapman–Enskog method, relaxation processes, completely ionized plasma, polar dielectrics and semiconductors, polarons, dielectrics.

1. Introduction

An arbitrary nonequilibrium state is described by the Liouville equation. Based on this equation, one can obtain kinetic equations in which a system is described by the one-particle distribution function (DF) $\mathrm{f}_{ap}(x,t)$ (here and in the following, the subscripts $a,b,c,...$ number the system components or the internal degrees of freedom of particles). This description becomes possible at times much greater than the characteristic time of particle collision τ_{col}. Kinetic equations are widely used in condensed matter theory. In the spatially homogenous case, the DF $\mathrm{f}_{ap}(x,t)$ is independent of the coordinates and describes relaxation processes in a narrow sense of this term.

Landau contributed considerably to their study with his local equilibrium concept [1], according to which the macroscopic parts of a system are described by equilibrium parameters and the corresponding DFs f_{ap}^{L}. According to Landau, such states are due to the fact that in the parts of the system that interact weakly with one another equilibrium is quickly established, and then relaxation processes take place. The chief drawback of this approach is that the DF f_{ap}^{L} is not an exact solution to the kinetic equation. A classical application of these ideas is the Landau theory of plasma component temperature relaxation [2]. Grad contributed considerably to the theory of relaxation processes too [3]. He postulated an expression for the DF in the form of the product of the locally equilibrium distribution f_{ap}^{L} and a truncated series expansion in the orthogonal Hermite polynomials. A drawback of his method is the absence of a small parameter that does not allow one to obtain the DF on the basis of the kinetic equation. An important application of these ideas is the Grad theory of Maxwellian relaxation (formation of dissipative hydrodynamic fluxes) [4].

A basis for the study of relaxation processes may be Bogolyubov's reduced description method (RDM), which is based on his idea of the functional hypothesis (FH) [5] (see also [6]). This idea is a generalization of the concept of normal solutions to the Hilbert kinetic equation, and it proposes a DF calculation procedure in a perturbation theory. As distinct from Hilbert, Bogolyubov substantiates the emergence of reduced description as a result of natural evolution. However, in this case, too, the absence of a small parameter is a problem. A small parameter allows one to calculate structures introduced by the FH and is the starting point for its proof (see, for example, [6]). Bogolyubov's RDM is a certain generalization of the Chapman–Enskog method (see, for example, [7]), which was intended for the study of spatially non-homogenous states with the aim to construct hydrodynamic equations. The topicality of refinement of today's theory of relaxation processes is pointed out in the literature [8, 9].

In this paper, relaxation processes are studied at their final stage. This introduces a small parameter, which is revealed when comparing a linear and a quadratic relaxation theory (although it already becomes obvious from dimensionality considerations). We were the first to formulate this problem [10, 11] and develop it in detail [12]. Emphasis is on considering relaxation processes in the vicinity of standard (single-fluid) hydrodynamics, although the theory allows one to describe processes in the vicinity of kinetic states too.

The paper, which is an overview of our results, is organized as follows. In Section 2, a general theory of relaxation processes at their final stage is constructed. Section 3 discusses the application of the theory to plasma physics and solid-state physics.

2. General theory of relaxation processes at their final stage

We study nonequilibrium processes in a certain system using a kinetic equation for the one-particle DF $\mathrm{f}_{ap}(x,t)$ (the subscript a numbers the internal degrees of freedom or system components). Collision integral $I_{ap}(\mathrm{f})$ of this equation is a functional of the DF f_{ap} as a function of the momentum p_l.

The main problem that is studied on the basis of the kinetic equation is the substantiation of the equations of standard (single-fluid) hydrodynamics, in which the system is described by the densities of additive integrals of motion, among which in most cases are the component particle number density $n_a(x,t)$, the energy density $\varepsilon(x,t)$, and the momentum density $\pi_l(x,t)$ (the variables $\zeta_\alpha(x,t)$). They satisfy the conservation laws in differential form which contain the particle number flux density $i_l(x,t)$, the energy flux density $q_l(x,t)$, and the momentum flux density $t_{nl}(x,t)$ (variables $\zeta_{\alpha l}(x,t)$). The conservation laws take place due to the properties of the collision integral which stem from the conservation of the number of particles, the energy, and the momentum, respectively, in collisions. Instead of the densities of the additive integrals of motion $\zeta_\alpha(x,t)$, standard hydrodynamics uses, together with the component particle number density $n_a(x,t)$, such quantities as the local temperature $T(x,t)$ and mass (drift) velocity $\upsilon_l(x,t)$ (parameters $\xi_\alpha(x,t)$). All mentioned variables are defined by the standard way [6] and are functionals of the DF $\zeta_\alpha(x,t) \equiv \zeta_\alpha(\mathrm{f}(x,t))$, $\zeta_{\alpha l}(x,t) \equiv \zeta_{\alpha l}(\mathrm{f}(x,t))$, $\xi_\alpha(x,t) \equiv \xi_\alpha(\mathrm{f}(x,t))$ ($\zeta_\alpha(\mathrm{f})$, $\zeta_{\alpha l}(\mathrm{f})$ are moments of the DF) .

A problem of great importance is the study of generalizations of hydrodynamics in which a system can be described by additional parameters $\theta_i(x,t) \equiv \theta_i(\mathrm{f}(x,t))$ (which are functionals of the DF f_{ap} and may be its moments too). The basis of our consideration, which relies on Bogolyubov's RDM, is Bogolyubov's idea of the functional hypothesis (FH) [6], which describes the nonequilibrium system DF structure at times much greater than some characteristic time τ_0

$$\mathrm{f}_{ap}(x,t) \xrightarrow[t \gg \tau_0]{} \mathrm{f}_{ap}(x,\xi(t),\theta(t)) . \tag{1}$$

Here $\mathrm{f}_{ap}(x,\xi,\theta)$ is a functional of the reduced description parameters (RDPs) $\xi_\alpha(x)$, $\theta_i(x)$ as functions of the coordinates x_l , which does not depend on the initial state of the system $\mathrm{f}_{ap}(x,t=0)$. The FH leads to a generalization of the Chapman–Enskog method, which is designed for constructing and studying equations of standard hydrodynamics [7]. The idea itself to seek for a solution of the kinetic equation in the form of the functional $\mathrm{f}_{ap}(x,\xi,\theta)$ in the case of standard hydrodynamics belongs to Hilbert; however, he did not set forth the idea that the natural evolution of a system results in the reduced description.

The kinetic equation (1) together with the FH allows one to express the time derivative of an arbitrary parameter in terms of the RDPs $\xi_\alpha(x,t)$, $\theta_i(x,t)$. In particular, it leads to closed equations of generalized hydrodynamics in the form

$$\partial_t \xi_\alpha(x,t) = L_\alpha(x,\mathrm{f}(\xi(t),\theta(t))) , \qquad \partial_t \theta_i(x,t) = M_i(x,\mathrm{f}(\xi(t),\theta(t))) , \tag{2}$$

where $L_\alpha(x,\mathrm{f})$, $M_i(x,\mathrm{f})$ are functionals known from the definitions of the RDPs.

The kinetic equation, the FH (1), and the time equations in (2) lead to the nonlinear integro-differential equation in the functional $\mathrm{f}_{ap}(x,\xi,\theta)$ named the kinetic equation at the reduced description (KERD). When solving this equation, one should take into account the RDP definitions $\xi_\alpha(\mathrm{f}(\xi,\theta)) = \xi_\alpha$, $\theta_i(\mathrm{f}(\xi,\theta)) = \theta_i$,which are called the additional conditions to KERD.

The traditional study of hydrodynamic states assumes a weak spatial inhomogeneity of the system, which is expressed by the estimates

$$\frac{\partial^s \xi_\alpha(x,t)}{\partial x_{l_1} ... \partial x_{l_s}} \sim g^s , \qquad \frac{\partial^s \theta_i(x,t)}{\partial x_{l_1} ... \partial x_{l_s}} \sim g^s \qquad (g \ll 1). \tag{3}$$

The small parameter g is estimated as $g = l_{\mathrm{fp}} / L$ where l_{fp} is the mean free path and L is the characteristic dimension of inhomogeneities in the system ($\partial / \partial x_l \sim L^{-1}$).

Unfortunately, the small parameter g does not suffice to solve KERD in a corresponding perturbation theory, which is termed the perturbation theory in RDP gradients. In this theory we have contributions $\mathrm{f}_{ap}^{(s)}$, $L_\alpha^{(s)}$, $M_i^{(s)}$ of the order g^s ($n \geq 0, L_\alpha^{(0)} = 0$) to DF $\mathrm{f}_{ap}(x,\xi,\theta)$ and the right sides of time equations (2).

KERD in the zeroth order in the gradients leads to the integro-differential equation with the additional conditions for contribution $\mathrm{f}_{ap}^{(0)}$

$$\sum_i \frac{\partial \mathrm{f}_{ap}^{(0)}}{\partial \theta_i} M_i^{(0)}(\mathrm{f}^{(0)}) = I_{ap}(\mathrm{f}^{(0)}), \quad \theta_i(\mathrm{f}^{(0)}) = \theta_i, \qquad \theta_i(\mathrm{f}^{(0)}) = \theta_i, \tag{4}$$

which are written in a somewhat schematic form. The nonlinear equation (4) with the additional conditions cannot be solved, thus giving no way to continue calculations in the perturbation theory. This equation describes processes that may take place in spatially homogenous states that in this paper are termed relaxation process in a narrow sense of this term (in a wide sense, every nonequilibrium process is a relaxation process). However, the parameters $\theta_i(x,t)$ can be chosen in such a way that in the usual hydrodynamic state (at $t \gg \tau_{\mathrm{fp}}$; τ_{fp} is free path time) they will be small quantity of the order ν. Then Eq. (5) can be sought in the form of a series in the powers of ν

$$\mathrm{f}_{ap}^{(0)} = w_{ap}\{1 + \sum_i A_{api}\theta_i + \sum_{ii'} B_{apii'}\theta_i\theta_{i'} + O(g^0\nu^3)\}, \tag{6}$$

where w_{ap} is the equilibrium DF, and A_{api}, $B_{apii'}$ are the functions of the parameters ξ_α to be found. This is the heart of our idea of generalization of the Chapman–Enskog method with the aim to study relaxation processes [10-12]. In this case, the time equation for the RDPs θ_i has the structure

$$\partial_t \theta_i = -\sum_{i'} a_{ii'}\theta_{i'} - \sum_{i'i''} b_{ii'i''}\theta_{i'}\theta_{i''} + O(g^0\nu^3, g^1), \tag{7}$$

where $a_{ii'}$, $b_{ii'i''}$ are some functions of the parameters ξ_α. We have proved that the A_{api}'s are linear combinations of the eigenfunctions of the operator of the collision integral linearized in the vicinity of the DF w_{ap}, and the $a_{ii'}$'s are linear combinations of the corresponding eigenvalues. So these quantities describe the kinetic modes of the system. In this general theory [12], we have studied the contributions $\mathrm{f}_{ap}^{(0,0)}$, $\mathrm{f}_{ap}^{(0,1)}$, $\mathrm{f}_{ap}^{(0,2)}$, $\mathrm{f}_{ap}^{(1,0)}$, and $\mathrm{f}_{ap}^{(1,1)}$ ($\mathrm{f}_{ap}^{(m,n)} \sim g^m\nu^n$) to the nonequilibrium DF f_{ap} and the generalized hydrodynamic equations (7) in the corresponding order of the perturbation theory.

Some inconvenience of this formulation of our theory lies in the fact that going to standard hydrodynamics from the equations (7) calls for an RDM based on the additional functional hypothesis that introduced the dependence $\theta_i^{\mathrm{h}}(x,\xi)$ of the parameters θ_i on the parameters ξ_α in standard hydrodynamics. To overcome this problem in our paper [12] it is proposed to consider variations $\delta\theta_i(x,t) \equiv \theta_i^{\mathrm{h}}(x,\xi(t)) - \theta_i(x,t)$ as new RDPs that obviously are small and to introduce a new small parameter μ by the estimate $\delta\theta_i \sim \mu$.

In Grad's well-known approach to the inclusion of relaxation processes into the theory of hydrodynamic phenomena [3, 4], the DF is postulated in the form

$$\mathrm{f}_{ap} = \mathrm{f}_{ap}^{L}\{1 + \sum_i h_{api}\theta_i\}, \qquad \int d^3p\, \mathrm{f}_{ap}\theta_{ip} = \theta_i \tag{8}$$

and is not calculated in a perturbation theory (see the assessment of the Grad method in [8]). In this case, the system is described by the parameters of the local equilibrium distribution and the additional parameters θ_i (the θ_{ip}'s are their microscopic values). The functions h_{api} are determined from the second equation in (8), and they are chosen in the form of a linear combination of the tensor Hermite polynomials $H_{n_1 \dots n_s}(\beta_a \vec{p})$, i.e. Grad's

DF in (8) is a truncated series in these polynomials (the factor β_a makes the polynomial argument dimensionless).

3. Application of the proposed theory

3.1. Maxwellian relaxation in a simple gas

Consider a system of identical particles with energy $\varepsilon_p = p^2/2m$ which in addition to the variables of standard hydrodynamics ξ_α (the particle number density n, the mass velocity υ_l, and the temperature T) is described by the additional parameters θ_i: q_l^o, π_{nl}^o, where q_l^o is the energy flux density and π_{nl}^o is the traceless momentum flux density in the accompanying coordinate system ($\pi_{nl} \equiv t_{nl} - \delta_{nl} t_{mm}/3$). After the free path time $\tau_{\rm fp}$, the quantities q_l^o, π_{nl}^o go into the dissipative fluxes $q_l^{o\rm h}$, $\pi_{nl}^{o\rm h}$, which describe the viscosity and the thermal conductivity of the system. It is the phenomenon of Maxwellian relaxation, whose theory was developed by Grad [3,4]. In this situation, the fluxes q_l^o, π_{nl}^o may be considered to be small quantities $\pi_{nl}^o \sim \nu nT$, $q_n^o \sim \nu nT(T/m)^{1/2}$ where ν is a new small parameter.

In the zeroth order in the gradients the system DF and time equation have structure similar to (6), (7). Obtained integral equations of the theory were sought for as truncated series in the orthogonal Sonine polynomials $S_n^\alpha(\beta\varepsilon_p)$ ($n = 0,1,2,...$; $\beta \equiv 1/T$) with an accuracy up to one and two polynomials. These polynomials are appropriate for the problem under consideration because they are orthogonal with the weight $w_p \varepsilon_p^{\alpha-1/2}$ (see, for example, [13]). It is shown that Grad's result for the damping coefficients λ_q, λ_π is given by our one-polynomial approximation [12] but the two-polynomial approximation gives corrections to them. In the Grad theory [3, 4], the nonequilibrium DF is not sought for in any perturbation theory; instead, it is finally written similar to (6). In the considered problem it gives the so-called 13-moment Grad approximation in the kinetics of hydrodynamic states of a simple gas. It is shown [12] that Grad's DF is given by the one-polynomial approximation in the solution of obtained by us integral equations. Our two-polynomial approximation refines Grad's result.

3.2. Two-fluid hydrodynamics of a completely ionized plasma

Let the hydrodynamic state of a completely ionized two-component plasma be described by the particle number densities n_a ($a = e,i$), the component mass velocities υ_{al}, and the component temperatures T_a. The evolution of spatially non-homogenous plasma is accompanied by the equalization of the component velocities and temperatures. On completion of the relaxation process, the plasma is described by standard (single-fluid) hydrodynamics, i.e. by the particle number densities n_a and the common mass velocity υ_l and temperature T. Our study is based on the Landau kinetic equation. The above-mentioned RDPs are defined by standard formulas (see, for example, [14,15]). According to the general theory, the relaxation may be studied using the variations $u_n = \upsilon_{ne} - \upsilon_{ne}^{\rm h}$, $\tau = T_e - T_e^{\rm h}$ of the electron variables υ_{en}, T_e from their values $\upsilon_{en}^{\rm h}$, $T_e^{\rm h}$ in the standard theory [16]. So the quantities u_n, τ play the role of the parameters $\delta\theta_i$ in the general theory and are considered to be small with a parameter μ, which is introduced by the estimates $\tau \sim \mu T$, $u_l \sim \mu(T / m_e)^{1/2}$. The ion variables υ_{in}, T_i are expressed in terms of the electron variables υ_{en}, T_e and the variables υ_n, T of standard hydrodynamics. Notice that the assumption of the smallness of the relative component velocity with the estimate $u_l \sim \mu(T / m_e)^{1/2}$ is widely used in the literature [14, 15] ($\upsilon_{ne} - \upsilon_{ni} \sim u_n$), however, without our construction of a systematic perturbation theory in the parameter μ. In the zeroth order in gradients, the system DF and the time equations may be written in the form (6), (7). The problem under consideration has a natural small parameter $\sigma = (m_e / m_i)^{1/2}$, which can be used in the solution of the integral equations of the theory. We analyze the contributions of a perturbation theory in σ up to terms of the next-to-the-leading order inclusive. They are calculated as truncated series in the Sonine polynomials, in each order in σ the consideration being restricted to the one-polynomial approximation [17, 18].

In the literature, two-fluid hydrodynamics is usually constructed on the basis of the local equilibrium assumption (see, for example, [14, 15]), according to which the zero approximation in gradients $\mathrm{f}_{ap}^{(0)}$ of the DF f_{ap} is given by the locally equilibrium distribution $\mathrm{f}_{ap}^{L} \equiv w_{ap-m_a\upsilon_a}(T_a)$ ($w_{ap}(T)$ is the Maxwell DF). In doing so, the contribution of the first order in gradients $\mathrm{f}_{ap}^{(1)}$ is sought for as a small correction to $\mathrm{f}_{ap}^{(0)} = \mathrm{f}_{ap}^{L}$. This assumption is an approximation, which may be termed the local equilibrium approximation (LEA). This approach stems from the idea that in the electron and ion subsystems equilibrium with different temperatures and velocities is quickly established, after which the subsystem temperatures and velocities equalize. This idea cannot be true because the function f_{ap}^{L} is not an exact solution of the kinetic equation.

The LEA cannot be considered as a simplified (modified) version of the Chapman–Enskog method. Only when a local equilibrium state is set up artificially can the LEA be considered as a theory of states in its vicinity. Well-known developments in this approach were made by Braginsky [14]; however, even his summarizing work [21] does not contain any systematic perturbation theory in gradients g and the small electron-to-ion mass ratio σ. Our study of the LEA applicability limits shows that only the leading contributions in the powers of σ to quantities in equations similar to (6), (7) taken in the one-polynomial approximation coincide with the LEA result. Next contributions of the theory refine this approximation. Notice that a detailed estimation of the role of these corrections calls for numerical calculations and depends on the ion charge.

We have also constructed a nonlinear (quadratic) relaxation theory [19], which is determined by the nonlinear terms in formulas of the type (6), (7). It is proved that even in the leading order in σ the LEA fails to give correct results. We have also shown that quadratic contributions to the time equations slow down the relaxation processes in the system at $\tau > 0$. We have studied the contributions $\mathrm{f}_{ap}^{(1,0)}$, $\mathrm{f}_{ap}^{(1,1)}$ to the DF of the first order in gradients $\mathrm{f}_{ap}^{(1)}$ ($\mathrm{f}_{ap}^{(1,s)} \sim g^1\mu^s$) and the corresponding contributions to the hydrodynamic equations with an additional analysis in a perturbation theory in the mass ratio σ [20]. The contribution $\mathrm{f}_{ap}^{(1,0)}$ leads to the standard hydrodynamics of the system, and the contribution $\mathrm{f}_{ap}^{(1,1)}$ refines its kinetic coefficients (defines their temperature dependence via τ) and introduces new dissipative processes that are due to the gradients of the variations u_l, τ. In particular, it is found that at $\tau > 0$ the plasma viscosity and thermal conductivity decrease.

3.3. Polaron hydrodynamics

Polar dielectrics or polar semiconductors are considered. In such systems electrons form a rarefied subsystem which interacts with the phonon subsystem. Interaction-dressed electrons are termed polarons. The investigation of non-equilibrium states of the polaron system is an important problem [22]. Interaction between electrons and phonons is described by the Froelich hamiltonian. Interaction of electron with optical phonons gives the main contribution to this interaction which has electrostatic nature and does not depend on the spin of an electron. For the investigation of the principal questions concerning the system, one usually neglects the band structure of the electron subsystem and chooses the energy of an electron in the form $\varepsilon_p = p^2 / 2m$. In this case the electron DF f_p summed up over the spin projections satisfies a kinetic equation with the usual structure and with the additional contribution of a weak constant electric field E_l. In our consideration the phonon subsystem is assumed to be in an equilibrium state with temperature T_0.

A kinetic equation for the electron subsystem was obtained by various researchers in the approximation of weak electron-phonon interaction. Without the assumption about weak inhomogeneity of the electron state a kinetic equation was obtained by one of the authors [23], which leads to a non-local collision integral $I_p(x,\mathrm{f})$. The usual collision integral takes into account the processes of emission and absorption of phonons by electrons. In these processes, the number of electrons is conserved but their momentum and energy are not conserved. Therefore, the conservation laws take additional terms that contain the electric field and sources of the energy and momentum.

The local temperature T and velocity υ_l of the polaron subsystem are defined by the standard formulas similar to ones used in the kinetic theory (see, for example, [6, 13, 15]). According to our general theory [12], the electron number density n belongs to the hydrodynamic parameters ξ_α, while the temperature T and the velocity υ_l belong to the relaxation ones θ_i. The standard hydrodynamic state of the electron subsystem is described by the diffusion equation [24], and relaxation processes are reduced to the equalization of the electron T and phonon T_0 temperatures and the establishment of the steady velocity $-\mu E_l$ in the presence of electric field (μ is the electron mobility). A small parameter ν that describes the system at the end of relaxation is chosen by the estimates $\tau \equiv T - T_0 \sim \nu T_0$, $\upsilon_l \sim \nu (m T_0)^{1/2}$, $E_l \sim \nu (n T_0)^{1/2}$, which already follow from dimension considerations (E^2 has the dimension of the energy density) .

In the zero approximation in gradients, the DF of the system and the time equations have the structure of the type (6) (7). Integral equations of the theory are solved by us by the method of truncated expansion in the Sonine polynomials with accuracy up to two polynomials inclusive [23]. In the literature the LEA is often used, which implies that the polaron DF f_p in the zero order in gradients $\mathrm{f}_p^{(0)}$ is given by the local equilibrium DF $\mathrm{f}_p^L \equiv w_{p-m\upsilon}(T)$ ($w_p(T)$ is the Maxwell DF; see, for example, [22]). Our consideration shows [23] that the LEA corresponds to the calculation of the quantities in formulas similar to (6), (7) in the one-polynomial approximation, and it is corrected in the two-polynomial approximation. But for additional terms in (6), (7) related to the electric field the LEA gives a wrong result even in the one-polynomial approximation.

Non-linear (quadratic) relaxation is investigated by us too. The coefficients in the time equations of the type (7) are calculated in the one- and the two-polynomial approximation and compared with the LEA results. Conditions under which the contributions of the quadratic relaxation are small in comparison with the contributions of the linear theory and restrictions for the small parameter ν are analyzed. For example, for high phonon temperatures T_0 compared to the Debye temperature T_D it was established the restriction for ν of the form $\nu \ll (T_D / T_0)^2 \ln(T_D / T_0)$.

Spatially non-homogenous states of the polaron subsystem are investigated based on the contributions $\mathrm{f}_p^{(1,0)}$, $\mathrm{f}_p^{(1,1)}$ to the DF of the first order in gradients $\mathrm{f}_p^{(1)}$ ($\mathrm{f}_p^{(1,s)} \sim g^1 \nu^s$) [25-27]. The corresponding contributions to the hydrodynamic equations are found and analyzed. It is noted that the phenomenon of momentum and energy transfer caused by the gradients of the polaron number n exists. The effect of relaxation processes and an external electric field on the kinetic coefficients of the system is analyzed including the viscosity and conductivity of the polaron subsystem. It is established on the basis of the non-local collision integral of the system obtained in [23] that it leads to small corrections in the electron-phonon interaction to the kinetic coefficients.

After the end of the temperature and velocity relaxation the system is described only by the density n, which satisfies the diffusion equation [24]. It was shown [28] that one can investigate these cases both on the basis of the kinetic equation and the hydrodynamic equations for the variables n, T, υ_l using our generalization of the Chapman–Enskog method.

3.4. Phonon hydrodynamics in dielectrics

The hydrodynamics of the phonon system of a dielectric is investigated. We restrict ourselves to the consideration of crystal lattices with the cubic symmetry. The quasi-momentum of a phonon p_l is changed within the Brillouin zone $\mathbf{B}$, and it do not conserve in the umklapp processes. The consideration is based on the kinetic equation for the phonon DF $\mathrm{f}_{\alpha p}(x,t)$ (the subscript α indicates the polarization) averaged over an elementary cell of the lattice. In an equilibrium state the DF $\mathrm{f}_{\alpha p}(x,t)$ does not depend on the coordinates, and in hydrodynamic states its gradients can be considered as small quantities. The kinetic equation takes into account at least inelastic pair collisions of phonons and the decay of a phonon into a pair of phonons; therefore, the number of phonons does not conserve in phonon processes. In this case, hydrodynamic equations follow from the energy conservation law and the momentum variation law with a source (friction force density) that is defined by the contribution of the umklapp processes to the collision integral. Hydrodynamic processes in the phonon system of a dielectric are described by the temperature T and the drift velocity υ_n. Their investigation is a classic problem of the solid state theory (especially in connection with the role of the umklapp processes in the conductivity of a dielectric). Usually one proceeds from the LEA (see, for example, [13, 29, 30]), which assumes that the hydrodynamic DF $\mathrm{f}_{\alpha p}$ in the zero-in-gradients approximation $\mathrm{f}_{\alpha p}^{(0)}$ is given by the local equilibrium distribution $\mathrm{f}_{\alpha p}^{L} = n_{\alpha p}$, where $n_{\alpha p} \equiv [e^{(\varepsilon_{\alpha p} - p_l \upsilon_l)/T} - 1]^{-1}$ is the shifted Planck distribution ($\varepsilon_{\alpha p}$ is energy of a phonon). According to Landau, this distribution gives standard definition of the drift velocity υ_n of the phonon system. The assumption of local equilibrium of the system at low temperatures is based on the idea that in these states the velocity varies slowly and at each value of the velocity the equilibrium described by the distribution $n_{\alpha p}$ has time to be established. However, this argument is not true because the distribution $n_{\alpha p}$ is not an exact solution of the kinetic equation.

Therefore, a general definition of the drift velocity υ_n is proposed by us with the help of an expansion of the average densities of phonon energy and momentum [31] in series in powers of υ_n which take into account the symmetry of the crystal. The corresponding small parameter μ defines the magnitude of the drift velocity by the estimate $\upsilon_n \sim \mu c$ where c is the velocity of the second sound in the dielectric. The higher terms in the expansion have a complicated tensor structure even for a crystal with cubic symmetry, which forces us to limit ourselves to the accuracy specified in the definition of υ_n. In the standard theory [13, 28, 29] the averages of energy and momentum densities are groundlessly calculated with the distribution $n_{\alpha p}$ that is in the LEA. In our theory these averages are calculated with the DF $\mathrm{f}_{\alpha p}$ to be found in a perturbation theory in the parameter μ. According to our general theory [12] the nonequilibrium DF and time equation for the velocity and temperature in the zero order in the gradients are given by formulas similar to (6), (7). Spatially non-homogenous states of the phonon system are investigated too based on the contributions $\mathrm{f}_{\alpha p}^{(1,0)}$, $\mathrm{f}_{\alpha p}^{(1,1)}$ to the DF of the first order in gradients $\mathrm{f}_{\alpha p}^{(1)}$ ($\mathrm{f}_{\alpha p}^{(1,s)} \sim g^1 \mu^s$) [31]. The corresponding contributions to the hydrodynamic equations are found and analyzed. It is noted that the phenomenon of friction depending on the temperature gradient and related to the umklapp processes exists (it was predicted first in the plasma physics [14]).

A further investigation of integral equations of the theory is unreal even for crystals of the cubic symmetry considered by us. There are certain possibilities at temperatures less than the Debye temperature $T < T_D$ because the contribution of the umklapp processes to the linearized collision integral is small in comparison with the contribution of the normal processes even at relatively low temperatures: it has the order $\lambda \equiv e^{-T_D/T}$ (see, for example, [6, 13]). Integral equations of our theory are solved by a simple iteration procedure in the powers of λ, which reduces them to Fredholm equations of the second kind that takes into account only normal collisions [31]. The investigation shows that the LEA is true only when using the standard definition of the drift velocity.

The solution of the obtained integral equations by the method of truncated expansion in orthogonal polynomials can be done only by simplifying the model. At $T \ll T_D$ kinetic processes in the phonon system are governed by long wave acoustic phonons [13, 29] and the exact theory can be approximated by some isotropic model based on the elasticity theory (see, for example, [30]). In this model energy of a phonon equals $\varepsilon_{\alpha p} = c_\alpha p$ and the calculation is made using the orthogonal polynomials $\Phi_n^s(\beta\varepsilon_{\alpha p})$ defined by the weight function $\varepsilon_{\alpha p}^s n_{\alpha p}^o (1 + n_{\alpha p}^o)$ where $n_{\alpha p}^o \equiv [e^{\varepsilon_{\alpha p}/T} - 1]^{-1}$ is the unshifted Planck distribution.

A classic problem of the dielectric theory is the problem of calculation of the heat conductivity for steady states of the system. Hydrodynamic equations in a steady state are solved by us in a perturbation theory in small gradients of the temperature. Finally, the expression for the heat conductivity in steady states κ_{st} is obtained [31] which shows that in the absence of the umklapp processes $\kappa_{\text{st}} = \infty$. One should expect this because steady states in an isolated dielectric exist only because of the umklapp processes. At low temperatures $T \ll T_D$ in the main approximation in λ and with the standard definition of the drift velocity our expression for κ_{st} gives [31] the well-known result by Akhiezer (see [6, 29]).

4. Conclusions

The Chapman–Enskog method of solution of kinetic equations is generalized to the case of the presence of relaxation processes in the system. The generalization is based on Bogolyubov's idea of the functional hypothesis. Linear and nonlinear relaxation processes are investigated for the first time at their final stage on the basis of a new small parameter.

The proposed theory of linear relaxation reduces to the solution of spectral problems for the operator of the linearized collision integral (that is, to Fredholm equations of the second kind). The convergence of the solutions of these equations obtained by the method of the truncated expansion in orthogonal polynomials needs a further investigation. The mentioned spectral problems describe the kinetic modes of the system. The standard Chapman–Enskog method reduces to the solution of a Fredholm integral equation of the first kind, and the convergence of the above-mentioned method in this case can be substantiated.

It is established that the local equilibrium assumption is only approximate. This leads to corrections, the significance of which needs a further numerical investigation.

As applications of the proposed theory, relaxation processes in well-known and important systems are investigated. These processes allow an experimental investigation because their description uses parameters that can be measured.

It is shown that the presence of additional small parameters of the theory (small mass ratio, low temperatures) can significantly simplify the investigation. In particular, the spectral problems of the theory can be reduced to the solution of Fredholm integral equations of the first kind.

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