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QUANTUM BOUND STATES ON THE NAKED REISSNER-NORDSTRÖM BACKGROUND

We present a quantum mechanical study of the stationary states for a test uncharged scalar field existing on the background of the Reissner-Nordström naked singularity in General Relativity. An analytical solution of the Klein-Gordon equation is obtained in terms of the confluent Heun function. The case, in which the wave function describes stable bound states, is examined. For this case, the equation that determines eigenenergies of the system is derived by matching the exact analytical solution and the appropriate asymptotical solution at infinity. This enables us to step up as compared to previous studies and to obtain some numerical results with the use of the confluent Heun function solution. Namely, the bound-state energy spectrum of the scalar field with sub-Planckian non-zero mass is calculated. The resulting eigenenergies of the Klein-Gordon equation turn out to be very close to those of the Dirac equation recently presented in literature. Also, the previously obtained Balmer-type formula for the eigenenergies is shown to give a good approximation in the case under consideration.

Keywords: Reissner-Nordström space-time, naked singularity, bound state, Klein-Gordon equation, confluent Heun function.

1. Introduction

In General Relativity it is usually thought that singularities of spacetime must be hidden inside black holes, instead of being “naked”, i.e. visible to distant observers. This is known as the weak cosmic censorship conjecture, which has not however yet been proven. Meanwhile, other studies [1] indicate that under certain circumstances naked singularities can appear as the result of a realistic gravitational collapse.

The naked singularities with electric charge attract much attention. They can act as particle accelerators in astrophysics. Besides, if one looks at particle physics from the viewpoint of General Relativity, for the electron the exterior geometry corresponds to the naked singularity because the electron charge-to-mass ratio exceeds a critical value.

The Reissner-Nordström (RN) spacetime is a solution to Einstein equations under the assumption of spherical symmetry with the point electric charge Q and mass M . It contains the naked singularity if $Q/M > 1$ (we use the geometrical units $G = \hbar = c = 1$).

In [2], for the naked RN singularity, we presented a quantum mechanical study of the bound states of an uncharged scalar field on the gravitational background. It is shown that there occur both the decaying bound states and the stable ones.

The main aim of the present work is to extend the investigation of the stable bound states on naked RN background by supplying an analytical solution to the Klein-Gordon equation and comparing its bound-state energy spectrum with that of the Dirac equation.

2. Solution to the Klein-Gordon equation

The line element of the RN geometry in the spherical coordinates is given by

$$ds^2 = Fdt^2 - F^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2),\ F = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \qquad (1)$$

The Klein–Gordon equation $(D^2 + m^2)\Psi = 0$ for an uncharged scalar field, or equivalently a test particle, of mass m on this background has stationary state solutions,which we decompose into the spherical harmonics

$$\Psi(t,r,\theta,\varphi) = \exp(-i\omega\, t) Y_{lm}(\theta,\varphi) \frac{1}{r\sqrt{F}} u(r)\,. \tag{2}$$

In what follows, we study the equation for the radial part of the wave function

$$\frac{d^2u}{dr^2} + \frac{1}{F}\left[\frac{1}{F}\left(\omega^2 - \frac{Q^2 - M^2}{r^4}\right) - m^2 - \frac{l(l+1)}{r^2}\right]u = 0. \tag{3}$$

We change the variable r and use the substitution

$$r = M + i\sqrt{Q^2 - M^2}(1-2z) \tag{4}$$

that shifts the singular points to their conventional locations at $z = 0$, $z = 1$ and $z = \infty$.

Upon inserting (4) into (3), it reduces to

$$\frac{d^2u}{dz^2} + \left(-\frac{\alpha^2}{4} + \frac{1-\beta^2}{4z^2} + \frac{1-\gamma^2}{4(z-1)^2} + \frac{1-2\eta}{2z} + \frac{2\delta - 1 + 2\eta}{2(z-1)}\right)u = 0 \tag{5}$$

where constants $\alpha, \beta, \gamma, \delta, \eta$ are expressed in terms of parameters Q, M, m and l.

Since in equation (5) singular points $z = 0$ and $z = 1$ are the regular ones whereas $z = \infty$ is the irregular singular point, this equation represents a form of the confluent Heun equation [4]. Its general solution obtained by means of computer algebra reads

$$\begin{aligned} u(z) &= \exp\left(\frac{1}{2}\alpha z + \frac{1}{2}(1+\beta)\ln z + \frac{1}{2}(1+\gamma)\ln(z-1)\right) \\ &\times \left[c_1 HeunC(\alpha,\beta,\gamma,\delta,\eta;z) + c_2 \exp(-\beta \ln z) HeunC(\alpha,-\beta,\gamma,\delta,\eta;z)\right] \end{aligned} \tag{6}$$

where $HeunC(\alpha,\beta,\gamma,\delta,\eta;z)$ is the confluent Heun function defined in the MAPLE package, c_1 and c_2 are integration constants. Note that equations (5) and (6) with different parameters describe also the scalar field in the Kerr-Newman spacetime [3]. However, in [3] only a qualitative analysis is presented. Thus the question whether one can obtain any numerical results expressing solutions in terms of the confluent Heun function remains open.

3. Description of bound states

In the present work we give a positive answer to the above-mentioned question and apply the confluent Heun function solution to calculate the bound-state spectrum of the Klein-Gordon equation on the RN naked singularity background.

First of all, it should be stressed that the time evolution of the scalar field on the naked RN background is non-unique. This means one has to specify an additional boundary condition at the singularity to obtain a fully unique dynamics. We adopt the Dirichlet boundary condition

$$u(z(r))\Big|_{r=0} = 0 \tag{7}$$

that corresponds to Friedrich's self-adjoint extension for the differential operator in (3). With this condition, only one of the integration constants in (6) remains independent being a normalization factor. As shown in [2], for the massive scalar field, there should exist stable bound states if the Q/M ratio exceeds a certain critical value (not to be confused with the $Q/M = 1$ value that separates the cases of the black hole and the naked singularity).

In order to calculate the bound-state eigenenergies, we employ the asymptotic behavior of the general solution (6) at infinity [4] and single out a decreasing solution

$$u_\infty(z) = z^{-\delta/\alpha}\exp(\alpha z/2) \tag{8}$$

where the exponential factor is damping because $\mathrm{Re}\,\alpha = 0$, $\mathrm{Im}\,\alpha > 0$ and $z \to +i\infty$ when $r \to +\infty$. Matching of the exact solution and the asymptotical one is easily implemented by using their wronskian

$$W(r) = u(z(r))\frac{du_\infty(z(r))}{dr} - u_\infty(z(r))\frac{du(z(r))}{dr},\ W(r_\infty) = 0 \tag{9}$$

with r_∞ denoting the numerical infinity – its value is adjusted to obtain stable numerical results. An example of application of this condition is given the next section.

4. Numerical results for eigenenergies

We are going to calculate the energy spectrum of the system with sub-Planckian masses in the sense that $m^2M^2 << 1$. In fact, this corresponds to the weak coupling limit.

In the case under consideration a good approximation to eigenenergies is provided by the Balmer-type formula derived with the use of the WKB method [2]

$$\omega_{WKB} = m\left[1 - \frac{m^2M^2}{(n + 1/2 + \sqrt{(l+1/2)^2 + m^2Q^2})^2}\right]^{1/2} \tag{10}$$

where n and l are, respectively, the radial and orbital quantum numbers. The derivation of this formula in [2] implies that the characteristic radius of the system is the Bohr radius $r_B = 1/(m^2M)$ that is confirmed in the present work by inspecting the plot of $u(r)$.

For the sake of brevity, we present only the results for eigenenergies. Computations were held using the values $Q = 2$, $M = 1$ and $m = 0.25$. Calculated relative binding energies $(m-\omega)/m$ are listed in Table 1.

Row entitled KG-Heun shows the results obtained in accordance with (9); in these computations the value of r_∞ was varied from $5r_B$ to $15r_B$. Row KG-WKB contains the results of (10). Row Dirac displays the results for the Dirac field in the same setup obtained by numerical integration in [5] (after being rescaled our values of Q, M and m correspond to Set III in Table 2 of [5]).

From Table 1, it is evident that there is only few percent difference between the bound-state energy spectra of the Klein–Gordon and Dirac equations.

Table 1

Relative binding energies $(m - \omega)/m$ for the Klein-Gordon and Dirac field bound states with quantum numbers (n, l). The parameters are $Q = 2$, $M = 1$, $m = 0.25$

Model	(n =0, l =0)	(n =1, l =0)	(n =2, l =0)	(n =0, l =1)	(n =1, l =1)
KG-Heun	$2.39\cdot 10^{-2}$	$7.04\cdot 10^{-3}$	$3.03\cdot 10^{-3}$	$7.64\cdot 10^{-3}$	$3.36\cdot 10^{-3}$
KG-WKB	$2.17\cdot 10^{-2}$	$6.44\cdot 10^{-3}$	$3.04\cdot 10^{-3}$	$7.24\cdot 10^{-3}$	$3.30\cdot 10^{-3}$
Dirac [5]	$2.3\cdot 10^{-2}$	$6.94\cdot 10^{-3}$	$3.23\cdot 10^{-3}$	$7.81\cdot 10^{-3}$	$3.5\cdot 10^{-3}$

5. Conclusions

In the work, we have studied an analytical solution for the scalar field on the naked RN background. An equation that determines the eigenenergies for the bound states of the system has been derived. The energy spectrum turns out to be close to that of the Dirac field. Therefore we can approximate the Dirac field by the simpler scalar one while considering dark-matter halos or particle-like configurations with naked singularities.

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