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CASIMIR EFFECT FOR DIRAC LATTICES

We consider polarizable sheets, which recently received some attention, especially in the context of dispersion interaction of thin sheets like graphene. These sheets are modeled by a collection of delta function potentials and resemble zero range potentials, known in quantum mechanics. We develop a theoretical description and apply the so-called TGTG-formula. Thereby we make use of the formulation of the scattering of waves off such sheets provided earlier.

Keywords: polarizable sheets, thin sheets, delta function potential.

1. Introduction

This paper is a contribution to the discussion of van der Waals and Casimir forces between surfaces. Last years, much attention was paid to the interaction between slabs of finite, and especially small, thickness. This triggered by a growing interest to two dimensional structures (sheets) like 2D el electron gas, monoatomically layers and graphene, as well as to the interaction etween them. As discussed in [1], the situation with sheets having only in-plane polarizability is relatively clear. Here one can formulate a hydrodynamic plasma model and calculate the quantities of interest [2]. The same holds for graphene described by the Dirac equation model for the π-electrons [3], which are responsible for the interaction with the electromagnetic field.

The situation with perpendicular polarizability is more complicated. While in [4] no response to the electromagnetic field was found, in [5] it was shown that such response take place. In that paper the sheet was modelled as a lattice of harmonic oscillators (dipoles), vibrating in direction perpendicular to the sheet, and the limit of zero spacing of this lattice was investigated. This was extended in [6] to a sheet, continuous from the very beginning, having parallel or perpendicular polarizabilities. Later, a similar setup was considered using point dipoles [7]. These can be represented by Dirac delta functions forming a so-called 'Dirac lattice'.

Such lattices, taken alone, have a well known internal dynamics, the simplest case being the Kronig-Penney model ('Dirac comb') [8]. In quantum mechanics their use is known as 'method of zero-range potential' [9]. As well known, in more than one dimension a Hamilton operator with a delta function potential is not self adjoint [10]. In electrodynamics, the self energy of a delta function potential is singular and one needs a renormalization. In terms of quantum field theory this setup was considered in [11].

Recently, a sheet of delta function potentials was used in [7] to model a polarizable sheet. For instance, scattering off such sheet was investigated and subsequently the transition to a continuous sheet as well.

In the present note we consider a setup of two such sheets, hold in parallel at some separation. This is a typical situation for the Casimir effect. It will allow not only to calculate the Casimir force, but also the limiting cases of a continuous sheet on one side and the dispersion interaction of two dipoles on the other side.

2. The vacuum energy

We use the following notations. A three dimensional vector is denoted by an arrow, a two dimensional vector parallel to the (x, y)-plane is denoted in bold. The lattice sites are given by $\mathbf{a_n} = a\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ where n_1 and n_2 are integers, a is the lattice spacing; the lattice is rectangular. The lattices A and B are given by

$$\vec{a}^{\mathrm{A}}_{(n)} = \begin{pmatrix} \mathbf{a}_n + \mathbf{c} \\ b \end{pmatrix}, \quad \vec{a}^{\mathrm{B}}_{(n)} = \begin{pmatrix} \mathbf{a}_n \\ 0 \end{pmatrix}, \tag{1}$$

The lattice B is in the (x, y)-plane and the lattice A is parallel to B on a separation b and shifted within the plane by $\mathbf{c}$.

For a single lattice we use the notations in [7] . The Greens function of a single lattice is given by eq.(85) there, which we use in the form

$$G(\vec{x}, \vec{x}\,') = G_0(\vec{x} - \vec{x}\,') - \sum_{n,n'} G_0(\vec{x} - \vec{a}_n)\phi^{-1}_{\mathbf{n}-\mathbf{n}'} G_0(\vec{a}_{\mathbf{n}'} - \vec{x}\,') \tag{2}$$

where

$$G_0(\vec{x} - \vec{x}\,') = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}-\vec{x}\,')}}{\omega^2 - \mathbf{k}^2 - k_3^2} = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}') + i\Gamma|\mathbf{x}_3 - x_{3'}|}}{2i\Gamma(\mathbf{k})}, \tag{3}$$

is the free Green function and we defined

$$\Gamma(\mathbf{k}) = \sqrt{\omega^2 - \mathbf{k}^2 + i0}. \tag{4}$$

The matrix $\phi^{-1}_{\mathbf{n}-\mathbf{n}'}$ is the solution of the equation

$$\sum_{\mathbf{k}} \left(\delta_{\mathbf{n},\mathbf{k}} + g G_0(\mathbf{a_n} - \mathbf{a}_k)\right)\phi^{-1}_{\mathbf{k}-\mathbf{m}} = g\,\delta_{\mathbf{n},\mathbf{m}}. \tag{5}$$

For a lattice with translational invariance, it is a function of the difference of the indices and it has the property (see eq.(103) in [7])

$$\sum_{\mathbf{m}} \phi^{-1}_{\mathbf{n}-\mathbf{m}} e^{i\mathbf{k}\mathbf{a_m}} = \frac{1}{\tilde{\phi}(\mathbf{k})} e^{i\mathbf{k}\mathbf{a_n}}, \tag{6}$$

where $\tilde{\phi}$ was derived in [7] , see eq. (109). For the scalar case it reads

$$\tilde{\phi} = \frac{1}{g_r} + \frac{i\omega}{4\pi} + J_1(\omega, k). \tag{7}$$

The renormalized coupling constant g_r was introduced and discussed in [7] . For two-dimensional lattice the function $J_1(\omega, k)$ is a double sum with omitted $\mathbf{n} = 0$ term,

$$J_1(\omega, k) = \sum_n{}' \frac{1}{|\mathbf{a}_n|} e^{i\omega|\mathbf{a}_n| + i\mathbf{k}\mathbf{a}_n}.$$

In general, the T-operator is defined by the symbolic relations $G = G_0 - G_0 T G_0$ (see, for example, eq. (10.29) in [12]). Comparing with (2) we get

$$T(\vec{x},\vec{x}') = \sum_{n,n'} \delta(\vec{x}-\vec{a}_n)\phi^{-1}_{\mathbf{n}-\mathbf{n}'}\delta(\vec{a}_{\mathbf{n}'}-\vec{x}'). \tag{8}$$

For the vacuum energy we use the TGTG-formula as given by eq. (10.40) in [12], which is the generalization of the Lifshitz formula,

$$E_0 = \frac{1}{2\pi}\int_0^\infty d\xi \operatorname{Tr}\ln(1-M), \tag{9}$$

and $\xi = -i\omega$ is the imaginary frequency. The kernel M is the product of two,

$$M(\vec{x},\vec{x}') = \int dx'' N_A(\vec{x},\vec{x}'') N_B(\vec{x}'',\vec{x}'), \tag{10}$$

and each factor is given by

$$N_{A,B}(\vec{x},\vec{x}') = \int dx'' G_0(\vec{x},\vec{x}'') T_{A,B}(\vec{x}'',\vec{x}'). \tag{11}$$

In this expression we insert (3) and (8) using $\vec{a}^{\mathrm{A}}_{(n)}$ from (1),

$$N_A(\vec{x},\vec{x}') = \sum_{\mathbf{n},\mathbf{m}}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{a}_\mathbf{n}-\mathbf{c})+i\Gamma|x_3-b|}}{2i\Gamma(\mathbf{k})}\varphi^{-1}_{\mathbf{n}-\mathbf{m}}\delta(\mathbf{a}_\mathbf{m}+\mathbf{c}-\mathbf{x}')\delta(b-x_{3'}). \tag{12}$$

Doing the substitution $\mathbf{n}\to\mathbf{n}+\mathbf{m}$ and using (6) , this expression can be rewritten,

$$N_A(\vec{x},\vec{x}') = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{c})+i\Gamma|x_3-b|}}{2i\Gamma(\mathbf{k})\tilde{\varphi}(\mathbf{k})}\sum_{\mathbf{m}} e^{-i\mathbf{k}\mathbf{a}_\mathbf{m}}\delta(\mathbf{a}_\mathbf{m}+\mathbf{c}-\mathbf{x}')\delta(b-x_{3'}). \tag{13}$$

The corresponding expression for $N_B(\vec{x},\vec{x}')$ can be obtained from (13) with $\mathbf{c}=0$ and $b=0$, as seen from (1) . Using these, we can write down the kernel $M(\vec{x},\vec{x}')$,

$$\begin{aligned} M(\vec{x},\vec{x}') = \int dx'' \int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{c})+i\Gamma|x_3-b|}}{2i\Gamma(\mathbf{k})\tilde{\phi}(\mathbf{k})}\sum_{\mathbf{m}} e^{-i\mathbf{k}\mathbf{a}_\mathbf{m}}\delta(\mathbf{a}_\mathbf{m}+\mathbf{c}-\mathbf{x}'')\delta(b-x_{3''}) \\ \cdot\int \frac{d^2\mathbf{k}'}{(2\pi)^2}\frac{e^{i\mathbf{k}'\mathbf{x}''+i\Gamma'|x_{3''}|}}{2i\Gamma(\mathbf{k}')\tilde{\phi}(\mathbf{k}')}\sum_{\mathbf{m}'} e^{-i\mathbf{k}'\mathbf{a}_{\mathbf{m}'}}\delta(\mathbf{a}_{\mathbf{m}'}-\mathbf{x}')\delta(x_{3'}). \end{aligned} \tag{14}$$

Carrying out the integration over $\vec{x}''$ using the delta function, we come to a sum over $\mathbf{m}$. Before doing this sum, we split the momenta $\mathbf{k}$ and $\mathbf{k}'$ into quasi momentum and integer part,

$$\mathbf{k} = \mathbf{q}+\frac{2\pi}{a}\mathbf{N}, \quad \mathbf{k}' = \mathbf{q}'+\frac{2\pi}{a}\mathbf{M}, \tag{15}$$

where $\mathbf{N}$ and $\mathbf{M}$ are integer vectors like $\mathbf{n}$ in $\mathbf{a}_\mathbf{n}$. The integration becomes $\int d^2\mathbf{k} = \int d^2\mathbf{q}\sum_\mathbf{N}$ and the components of $\mathbf{q}=(q_1,q_2)$ are restricted to $0\le q_{(1,2)} < 2\pi/a$. Now the sum appearing in (14) gives

$$\sum_{\mathbf{m}} e^{-i(\mathbf{k}-\mathbf{k}')\mathbf{a}_{\mathbf{m}}} = \left(\frac{2\pi}{a}\right)^2 \delta^{(2)}(\mathbf{q}-\mathbf{q}'). \tag{16}$$

The dependence on $\mathbf{N}$ and $\mathbf{M}$ drops out. Then (14) turns into

$$M(\vec{x},\vec{x}') = \frac{1}{a^2}\int\frac{d^2\mathbf{q}}{(2\pi)^2}\sum_{\mathbf{N}}\frac{e^{i\mathbf{x}(\mathbf{q}+\frac{2\pi}{a}\mathbf{N})-i\frac{2\pi}{a}\mathbf{Nc}+i\Gamma|x_3-b|}}{2i\Gamma(\mathbf{k})\tilde{\varphi}(\mathbf{k})}\sum_{\mathbf{M}}\frac{e^{i\frac{2\pi}{a}\mathbf{Mc}+i\Gamma'|b|}}{2i\Gamma(\mathbf{k}')\tilde{\varphi}(k')}\sum_{\mathbf{m}'}e^{-i\mathbf{q}\mathbf{a}_{\mathbf{m}'}}\delta(\mathbf{a}_{\mathbf{m}'}-\mathbf{x}')\delta(x_{3'}), \tag{17}$$

This expression for $M(\vec{x},\vec{x}')$ must be inserted into the Lifshitz formula (9) under the sign of the trace. In doing so, the delta function in one factor $M(\vec{x},\vec{x}')$ turns the $\mathbf{x}$ in the next factor into $\mathbf{a}_{\mathbf{m}'}$. Again, the summation over $\mathbf{m}'$ gives a delta function for the quasi momenta. In this way, the products of the factors $M(\vec{x},\vec{x}')$ is diagonal in q and we can define

$$h(\omega,\mathbf{q}) = \frac{1}{a^2}\sum_{\mathbf{N}}\frac{e^{i\Gamma(k)b+i\frac{2\pi}{a}\mathbf{Nc}}}{2i\Gamma(\mathbf{k})\tilde{\varphi}(\mathbf{k})} \tag{18}$$

with $\mathbf{k}$ given by (15) , and rewrite (17) in the form

$$M(\vec{x},\vec{x}') = a^2\int\frac{d^2\mathbf{q}}{(2\pi)^2}e^{i\mathbf{qx}}h(\omega,\mathbf{q})h(\omega,\mathbf{q})^*\sum_{\mathbf{m}}e^{-i\mathbf{q}\mathbf{a}_{\mathbf{m}}}\delta(\mathbf{a}_{\mathbf{m}}-\mathbf{x}')\delta(x_{3'}). \tag{19}$$

In this way, the factors $M(\vec{x},\vec{x}')$ entering (9) become diagonal. The $\mathbf{x}$-integration in the trace is to be taken over one cell and it turns the sum into unity,

$$\int d\vec{x}\, e^{i\mathbf{qx}}\sum_{\mathbf{m}}e^{-i\mathbf{q}\mathbf{a}_{\mathbf{m}}}\delta(\mathbf{a}_{\mathbf{m}}-\mathbf{x}')\delta(x_{3'}) = 1. \tag{20}$$

As a consequence, for the vacuum energy per cell we get the expression

$$E_0 = \frac{a^2}{2\pi}\int_0^\infty d\xi\int\frac{d^2\mathbf{q}}{(2\pi)^2}\ln\left(1-h(i\xi,\mathbf{q})h(i\xi,\mathbf{q})^*\right). \tag{21}$$

This is the final general formula for the vacuum energy. It is convergent for the same reasons which usually apply to the Lifshitz formula and it can be evaluated numerically.

Similarly, one dimensional delta lattices (chains) in 3D may be considered. In this case the chains A and B are given by

$$\vec{a}^{\mathrm{A}}_{(n)} = \begin{pmatrix} a_n + c \\ b \\ 0 \end{pmatrix}, \quad \vec{a}^{\mathrm{B}}_{(n)} = \begin{pmatrix} a_n \\ 0 \\ 0 \end{pmatrix}. \tag{22}$$

The chain B is placed along the x_1 axis, the chain A is parallel to it at the distance b, its lattice sites being shifted by c. In this setup we may study the Casimir normal and lateral forces.

The vacuum energy per one unit is given by

$$E_0 = \frac{a}{2\pi}\int_0^\infty d\xi \int_0^{2\pi/a} \frac{dq}{2\pi} \ln\left(1 - h(i\xi, q) h(i\xi, q)^*\right). \tag{23}$$

With

$$h(i\xi, q) = \frac{1}{2\pi a} \sum_{N=-\infty}^{\infty} \frac{e^{i\frac{2\pi}{a}Nc}}{\tilde{\phi}(k)} K_0(\sqrt{k^2+\xi^2}b), \quad k = q + \frac{2\pi}{a}N. \tag{24}$$

In the function $\tilde{\phi}(k)$ defined by (7) the sum is computed exactly

$$J_1(i\xi, k) = \sum_n{}' \frac{e^{ia(i\xi|n|+kn)}}{a|n|} = -\frac{1}{a}\ln\left(1 + e^{-2a\xi} - 2\cos(qa)e^{-a\xi}\right).$$

3. Conclusions

We considered T- matrix operator for a lattice of δ-functions in terms of the function involving a lattice sum, which can be expressed via Hurwitz zeta function. Further we used this T-operator to formulate the kernel of the TGTG formula which is in fact a generalization of the Lifshitz formula. In this way a finite (converging) expression for the interaction energy is found. It can be specified for the interaction of 2-dimensional lattices and, as well, of 1-dimensional lattices (chains). Also it allows to consider the translation of one lattice parallel to another and the rotation. This is left for future work.

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