UDC 539.12.01

V. I. Demchik, A. V. Gulov, V. V. Skalozub*

Oles Honchar Dnipropetrovsk National University, Dnipropetrovsk, Ukraine
**e-mail: skalozub@dsu.dp.ua*

SPONTANEOUS SYMMETRY BREAKING IN THE O(4) SCALAR MODEL ON A LATTICE

In the present paper we investigate the spontaneous symmetry breaking in the four-component scalar $\lambda\varphi^4$ model on a lattice (the O(4) model) for the wide interval of coupling constant values. We consider the case of zero temperature and the four-dimensional space-time. The massless Goldstone components of the scalar field are integrated out in the spherical coordinates in the internal space of the scalar field by the saddle point method, and the initial functional integral of the model is reduced to an effective one-component theory with massive radial component convenient for lattice investigations. The model is expressed in terms of dimensionless parameters suitable for varying coupling constant and rescaling of the lattice space and the mass. Monte Carlo simulations are performed with the QCDGPU software package on a HGPU cluster. It is shown that the symmetry is spontaneously broken for coupling values $\lambda>\lambda_0 \sim 10^{-5}$. For smaller coupling values, the scalar field vanishes on a lattice, which can be interpreted as instability of the homogeneous condensate or even instability of the model itself. The critical value of coupling λ_0 is independent of the lattice sizes for investigated lattices 16^4 and 32^4.

Keywords: scalar model, Monte Carlo simulations, spontaneous symmetry breaking.

У роботі досліджується спонтанне порушення симетрії в чотирьохкомпонентній скалярній $\lambda\varphi^4$ моделі на ґратці (O(4) моделі) в широкому інтервалі значень константи взаємодії. Розглядається випадок нульової температури та чотиривимірного простору-часу. Безмасові компоненти Голдстоуна скалярного поля проінтегровано у сферичних координатах у внутрішньому просторі скалярного поля за допомогою методу перевалу, а початковий функціональний інтеграл моделі приведено до ефективної однокомпонентної теорії з масивною радіальною компонентою, яка є зручною для досліджень на ґратці. Модель представлено в термінах безрозмірних параметрів, придатних для зміни значень константи взаємодії, масштабу ґратки та маси. Проведено симуляції Монте Карло на кластері HGPU за допомогою програмного забезпечення QCDGPU. Показано, що симетрія є спонтанно порушеною при значеннях константи взаємодії $\lambda > \lambda_0 \sim 10^{-5}$. Для менших значень константи взаємодії скалярне поле на ґратці зникає, що можна пояснити як нестабільність однорідного конденсату або навіть нестабільність самої моделі. Критичне значення взаємодії λ_0 не залежить від розміру ґраток 16^4 та 32^4, що досліджувалися.

Ключові слова: скалярна модель, симуляції Монте Карло, спонтанне порушення симетрії.

В работе исследуется спонтанное нарушение симметрии в четырехкомпонентной скалярной $\lambda\varphi^4$ модели на решетке (O(4) модели) в широком интервале значений константы взаимодействия. Рассматривается случай нулевой температуры и четырехмерного пространства-времени. Безмассовые компоненты Голдстоуна скалярного поля проинтегрированы в сферических координатах во внутреннем пространстве скалярного поля при помощи метода перевала, а исходный функциональный интеграл модели приведен к эффективной однокомпонентной теории с массивной радиальной компонентой, которая удобна для исследований на решетке. Модель представлена через безразмерные параметры, пригодные для изменений значений константы взаимодействия, масштаба решетки и массы. Проведены симуляции Монте Карло на кластере HGPU при помощи программного обеспечения QCDGPU. Показано, что симметрия спонтанно нарушена при значениях константы взаимодействия $\lambda > \lambda_0 \sim 10^{-5}$. Для меньших значений константы взаимодействия скалярное поле на решетке исчезает, что можно объяснить нестабильностью однородного конденсата или даже нестабильностью самой модели. Критическое значение взаимодействия λ_0 не зависит от размера решеток 16^4 и 32^4, которые исследовались.

Ключевые слова: скалярная модель, симуляции Монте Карло, спонтанное нарушение симметрии.

1. Introduction

Symmetry breaking and phase transitions in the O(N) scalar models are the problems of great importance in quantum field theory. They are investigated in numerous publications for different number of components N and different space-time dimensions d = 2, 3, 4 by using various methods of calculations, in particular, Monte Carlo (MC) simulations on a lattice [1-3]. In what follows, we fix the space-time dimensionality d = 4 and consider the models at zero temperature (symmetric lattices). The numbers of components N = 1 and N > 1 correspond to qualitatively different physics related with spontaneous breaking of discrete and continuous symmetry. The choice N = 4 has relevance to either the One-Higgs-Doublet standard model of particles or low energy limits of QCD and its phase structure at finite temperature.

Recently, it was discovered in MC simulations for the O(1), d = 4 model that the type of the phase transition at finite temperature depends on the value of coupling constant λ [4]. This phenomenon had not been discussed in the literature before. Usually, the coupling is assumed to be small and has the order 0.01 – 0.1, whereas the type of the phase transition changes at much lower values of the coupling. Naturally, this problem is of interest for other O(N) models. However, for $N\geq 2$ computations become much more complicated in comparison with the one-component model.

In the present paper, we investigate spontaneous symmetry breaking (SSB) in the O(4) model on a lattice at zero temperature and its dependence on the value of λ. The first main problem is a correct treatment of the Goldstone modes. These modes are related with spontaneous breaking of continuous internal symmetries. So, it is impossible to realize them on a lattice exactly, only some remnants can be observed. The second problem is to work out a procedure for MC simulations which can be efficient for an extremely wide interval of coupling values. To clarify the first problem, we consider the behavior of the partition function (PF) in the continuous theory in case when the homogeneous condensate of scalar field forms the background for quantum fluctuations. We develop a general approach for solving this problem by using the representation of the PF in spherical coordinates in the internal space of scalar fields. The main idea is to integrate out the continuous angular modes, which become Goldstone bosons after symmetry breaking, before lattice investigations. To solve the second problem, we introduce special dimensionless variables.

The integration over angular Goldstone's modes is carried out by saddle-point integration in the spherical coordinates and the effective potential for the radial (Higgs's) field derived. With this effective action, either symmetry breaking or its dependence on λ value are investigated in the way similar to the O(1) case [4]. We perform the MC simulations for the radial field and investigate how the SSB depends on the value of coupling λ. We discuss the results obtained and prospects for future studies.

2. The scalar model in spherical coordinates

The Lagrangian of the model reads

$$L=\frac{1}{2}\frac{\partial\varphi}{\partial x_\mu}\frac{\partial\varphi}{\partial x_\mu}-V(\varphi),\quad V(\varphi)=-\frac{m^2}{2}\varphi^2+\frac{\lambda}{4}\varphi^4 \tag{1}$$

where $\varphi^2=\sum_{i=1,\dots,4}\varphi_i^2$. For $m^2>0$ the SSB potentially takes place.

Let us parameterize the scalar field in the spherical coordinates as $\varphi_i(x) = R(x)n_i(x)$, where n_i is a unit vector in the internal space. The direction n_i contains angular variables describing a point on the four-dimensional sphere,

$$n_1 = \sin\theta_1 \sin\theta_2 \sin\theta_3, \quad n_2 = \sin\theta_1 \sin\theta_2 \cos\theta_3, \quad n_3 = \sin\theta_1 \cos\theta_2, \quad n_4 = \cos\theta_1. \qquad (2)$$

In these variables, the Lagrangian (1) has the form

$$L = \frac{1}{2}\frac{\partial R}{\partial x_\mu}\frac{\partial R}{\partial x_\mu} + \frac{R^2}{2}\sum_i \frac{\partial n_i}{\partial x_\mu}\frac{\partial n_i}{\partial x_\mu} - V(R). \qquad (3)$$

The Euler-Lagrange equations of the model have the constant solution $R_c = m/\sqrt{\lambda}$, n_i =const called the classical condensate. In the vicinity of this solution, fluctuations of n_i describe massless particles corresponding to the Goldstone modes appearing after symmetry breaking, whereas fluctuations of *R* correspond to massive particles.

When the Cartesian coordinates are used in the internal space, the functional integrals in the model are of the Gaussian type for free particles. In the spherical coordinates, the Jacobian appears as an additional factor in the integrand. This requires developing a procedure for calculating such integrals.

The PF of the model in terms of spherical coordinates in the internal space reads

$$Z = \int D\varphi \exp\left(\int d^4x\, L\right) = \int_0^\infty R^3 DR(x) \int d^3\Omega(x) \exp\left(\int d^4x\, L\right), \qquad (4)$$

where $d^3\Omega(x)$ represents the integration over angular variables in the internal space. We assume a lattice in the Euclidian space-time with the spacing *a*. The field *R*(*x*) is determined by its values in each site *x* of the lattice. In what follows, we consider zero temperature and use symmetric lattices.

We consider the classical condensate as the background for quantum fluctuations and calculate the corresponding PF. A usual procedure in this case is to calculate the effective potential $V(\varphi_c)$ and apply the saddle-point method. For constant fields, the equations to find the stationary point are simplified significantly, and the integration in the functional integral can be easily fulfilled.

The PF can be written in the form

$$Z = \prod_x \int_0^\infty DR(x) \int d^3\Omega(x) \exp\left(3\log R(x) + a^4 L(x)\right) \qquad (5)$$

The "effective potential" for this case is

$$\tilde{V}(R) = \sum_x a^4 V(R(x)) - \sum_x 3\log R(x). \qquad (6)$$

In fact, this is not a usual effective potential obtained by the Legendre transformation but a useful tool for calculation by the saddle-point method. The stationary equation, $\partial\tilde{V}(R)/\partial R = 0$, has one real solution

$$R_0^2 = \frac{m^2}{2\lambda}\left(1 + \sqrt{1 + \frac{12\lambda}{m^4 a^4}}\right). \qquad (7)$$

For small λ, it can be written as $R_0^2 = m^2\lambda^{-1} + 3m^{-2}a^{-4}$. The effective potential in the vicinity of R_0^2 equals to

$$\tilde{V}(R) = \tilde{V}(R_0) + \frac{1}{2}\left(\frac{12\lambda}{m^2} + \frac{3}{m^2a^4}\right)(R(x) - R_0)^2 + \dots . \tag{8}$$

Our next step is to calculate the PF in the spherical coordinates in some way. The goal is to take into consideration the contributions of the continuous Goldstone modes analytically. After that, the radial field $R(x)$ remains the only dynamical variable. In this way we obtain an effective theory convenient for further lattice investigations. In what follows, we will use the saddle-point approach for integration. Its leading approximation accounts for the non-analytic correlations existing between the parameters of the problem. Remaining contributions can be calculated in perturbation theory.

3. Saddle-point integration in the spherical coordinates

Now, we carry out angular integration in the internal space of the model. The functions n_i are given in (2). For the second term in (3) we get

$$L(\theta_i) = \frac{R^2}{2}\left[\left(\frac{\partial\theta_1}{\partial x_\mu}\right)^2 + \left(\frac{\partial\theta_2}{\partial x_\mu}\right)^2 \sin^2\theta_1 + \left(\frac{\partial\theta_3}{\partial x_\mu}\right)^2 \sin^2\theta_1 \sin^2\theta_2\right] \tag{9}$$

and the Jacobian reads $J(\theta_i) = R^3 \sin^2\theta_1 \sin\theta_2$.

To calculate the PF, we construct the "effective potential" (6), determine the stationary point for the fixed direction $R = R_0$, $\theta_i = \theta_i^{(0)}$, and write the integration variables as $R(x) = R_0 + \rho(x)$, $\theta_i(x) = \theta_i^{(0)} + \eta_i(x)$, where i = 1,2,3. After that, we can integrate over angular variables $\eta_i(x)$ by using the saddle-point method.

Substituting the angles in (9) and in the Jacobian by means of $\sin\theta_i(x) = \sin\theta_i^{(0)} + \cos\theta_i^{(0)}\eta_i(x) + O(\eta_i^2)$, we expand them in series over $\eta_i(x)$. In the given approximation, the angular term (9) becomes

$$L(\theta_i) = -\frac{R_0^2}{2}\left[\eta_1\frac{\partial^2}{\partial x_\mu^2}\eta_1 + \sin^2\theta_1^{(0)}\eta_2\frac{\partial^2}{\partial x_\mu^2}\eta_2 + \sin^2\theta_1^{(0)}\sin^2\theta_2^{(0)}\eta_3\frac{\partial^2}{\partial x_\mu^2}\eta_3\right] + \dots \tag{10}$$

where we omit interaction terms with three and more fields. Then, we can extend the integration limits for $\eta_i(x)$ to plus-minus infinities and make the change of variables: $r_1 = R_0a^2\eta_1$, $r_2 = R_0a^2\eta_2\sin\theta_1^{(0)}$, $r_3 = R_0a^2\eta_3\sin\theta_1^{(0)}\sin\theta_2^{(0)}$. After these transformations, we obtain the PF measure (4):

$$R^3(x)\sin^2\theta_1(x)\sin\theta_2(x)DR\,D\theta_1\,D\theta_2\,D\theta_3 = \frac{R^3(x)}{R_0^3a^6}D\rho\,Dr_1\,Dr_2\,Dr_3 \tag{11}$$

and the angular exponential contains $-(1/2)\sum_i r_i\,\partial^2/\partial x_\mu^2\,r_i$. Due to the symmetry of the integration limits, the linear in $\eta_i(x)$ terms do not contribute to the result. Thus, we obtain three integrals over the angular variables. These integrals are independent of any physical variables and, after extending the limits of integration to infinities, result in some

constant factor. This factor does not influence the PF behavior in the limit $\lambda \to 0$. Another property of (11) is independence of the r.h.s. of the arbitrary saddle point parameters $\theta_i^{(0)}$. The dependence is cancelled, as it is expected. Hence, we conclude that the SSB of continuous symmetry can be completely related with the radial variable $R(x)$. After angular integrations, we can integrate over the radial field ρ.

The factor $R^3(x)/R_0^3$ in (11) can be transformed to the extra term in the "effective action" (6) for the one-component field $R(x)$. This observation gives a possibility for formulating a general procedure for investigations of SSB in the O(N) models. Namely, we can start with the effective one-component Lagrangian consisting of the initial one written in the spherical coordinates where we can omit the angular terms and add the term $\log(R(x)/R_0)^{N-1}$. The value of the saddle point R_0 has to be calculated from the "effective potential" $\tilde{V}(R)$ for fixed angular variables. The procedure of dealing with the one component field $R(x)$ can be simply realized on lattices similarly to the O(1) case [4].

4. Monte Carlo lattice simulations

The phase with the broken symmetry in the O(N) model at zero temperature on a lattice can be described by the Euclidian effective one-component Lagrangian consisting of the radial part of the initial action and the additional term $\log(R/R_0)^{N-1}$ from the Jacobian and integrated angular part:

$$S = \sum_x a^4 \left(\frac{1}{2} \frac{\partial R}{\partial x_\mu} \frac{\partial R}{\partial x_\mu} - \frac{m^2}{2} R^2 + \frac{\lambda}{4} R^4 \right) - \sum_x \log(R/R_0)^{N-1} \tag{12}$$

where a is the lattice spacing, and the value R_0 is calculated from the "effective potential" $\tilde{V}(R)$ for fixed angular variables. For MC simulations, we use a symmetric hypercubic lattice with hypertorous geometry in the four-dimensional space-time.

The one-component non-negative scalar field $R(x)$ is defined in lattice sites. As it is known, symmetric $d = 4$ lattice corresponds to zero temperature, whereas finite temperatures require a less number of sites in the temporal direction than in the spatial directions.

Since we are interested in varying λ in a wide interval of values, self-adjustment of the lattice spacing and other parameters of the model is of great importance. In this regard, we rewrite the action through dimensionless quantities. The classical condensate value $R_c = m/\sqrt{\lambda}$ is set as a unit of field. Writing the radial field as

$$\frac{R(x)}{R_c} = \sqrt{2\Phi_x}\,, \tag{13}$$

we obtain the lattice action (up to a constant)

$$S = \frac{1}{z} \sum_x \Phi_x \left(\Phi_x - 1 + \frac{1}{4} \sqrt{\frac{z}{\lambda}} \sum_\mu \log^2 \frac{\Phi_{x+a\hat{\mu}}}{\Phi_x} \right) - \sum_x \frac{N-1}{2} \log \Phi_x \tag{14}$$

where the log-squared term encodes the kinetic part of the action written through the finite differences instead of derivatives, and z is a dimensionless parameter defined as

$$z = \frac{\lambda}{m^4 a^4}. \qquad (15)$$

The effective model (12) is derived by integration of angular degrees around the condensate direction. This means quantum fluctuations have to be not larger than the values of the condensate. So, the values $R >> R_c$ could be incompatible with the effective model. Fortunately, the probability of such large values decreases as $\exp\left(-\Phi^2/z\right) \approx \exp\left(-R^4/(4zR_c^4)\right)$, $R \to \infty$, in accordance with (14). In this regard, the interval of values of R appears to be actually finite and can be cut by some upper bound $R_{\max}$. In MC simulations, we use two values $R_{\max} = 3.75R_c$ and $R_{\max} = 4R_c$ to show that the results are independent of the cutoff scale. In the selected interval, R is taken to be uniformly distributed in accordance to the definition of the partition function of the effective theory. The chosen cuts correspond to probability < 10^{-22} (the choice of z is described below), so the obtained Boltzmann ensembles demonstrate no essential dependencies on $R_{\max}$.

Parameter z is related to the saddle point position R_0 defined in (7), $2\left(R_0/R_c\right)^2 = 1 + \sqrt{1 + 4(N-1)z}$. We use $z \approx 1$ ($z = 0.67$ and $z = 1$, actually) corresponding to saddle point position at the central part of the interval of field values, $1 < R_0/R_c \leq 1.5$. Extremely small values of z could make the N-dependent term in the effective action negligible with respect to other terms, whereas large z could move the saddle point outside the cutoff scale.

In this paper we consider MC simulations at $N = 4$, only. Other possible values of N we tried lead to similar results. All the calculations are performed with the double precision. The system is thermalized by passing up to 10^5 MC iterations for every run. For measuring, we use 10^3 MC configurations separated by 10 bulk updates. To exclude lattice finite-size effects we perform simulations on lattices 16^4 and 32^4. The full correspondence of key results is obtained on these different lattices.

For MC simulations, we use the universal software environment QCDGPU [5] that allows performing simulations for a number of frequently studied models. To produce pseudo-random numbers in QCDGPU package we use our own library of pseudo-random number generators for MC simulations PRNGCL [6].

All the simulations are performed on a heterogeneous distributed HGPU cluster [7]. It consists of different graphics processing units: AMD Radeon HD 7970, HD 6970, HD 5870, HD 5850, HD 4870, HD 4850, NVIDIA GeForce GTX 560 M, GTX 560 Ti.

The dependence of the averaged radial field R/R_c on the coupling constant λ at zero temperature on the symmetric lattice 16^4 is shown in Fig. 1. Let us note that this quantity is not the scalar condensate value (the order parameter). In fact, it accounts for the scalar field coming from the classical condensate, quantum fluctuations and the part due to the logarithm of the Jacobian. The latter part is independent of the sign of the mass squared. In the case of classical condensate with no fluctuations, $R/R_c = 1$ independently of λ. As we can see, R/R_c is close to 1 for popular values of $\lambda \geq 10^{-2}$, the quantum correction to this quantity is no more than 40%. However, R/R_c goes to zero at extremely small couplings. At $\lambda < \lambda_0 \approx 10^{-5}$ it is of one order less than the classical condensate value. At extremely small couplings the scalar field on a lattice exhibits tendency to be negligibly small with respect to the value of classical condensate. This result contradicts the idea of small quantum fluctuations over a strong homogeneous condensate used in our calculations. The same value of λ_0 is also detected on the lattice 32^4 that demonstrates the

stability of the results with respect to finite-volume effects. This dependence is not expected at the classical level where the condensate of φ has the value $|m|/\sqrt{\lambda}$.

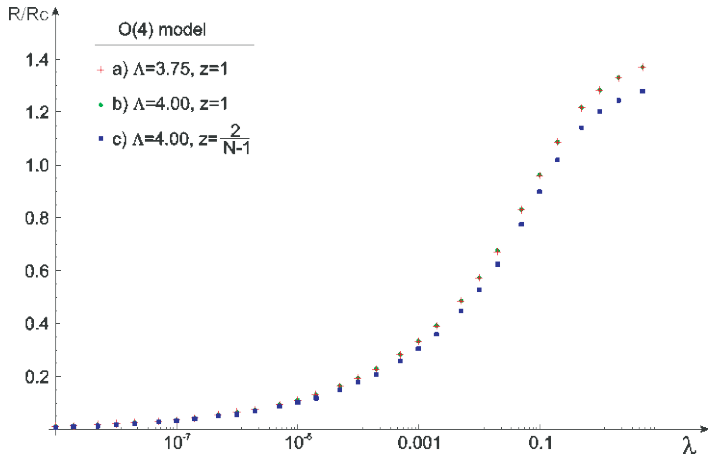

Fig. 1. Dependence of the mean radial field (in the classical condensate units) on the coupling constant λ at zero temperature on a lattice 16^4.

We cannot interpret the decay of the homogeneous condensate on a lattice at $\lambda \to 0$ as some kind of symmetry restoration. First, there is no evident saddle point to integrate the angular modes in spherical coordinates in the case of restored symmetry, so, the restored phase is probably outside the scope of our effective model. Second, the classical potential becomes unstable at $\lambda = 0$. In this regard, we can assume the instability of the homogeneous condensate or even instability of the O(N) model itself at $\lambda \approx 10^{-5}$.

The change of the SSB at extremely low couplings reminds in some aspects another phenomenon already found for continuous field theory by Linde [8] and Weinberg [9] in the Higgs model and in the standard model, respectively. These authors observed that the SSB does not happen for small values of the coupling $\lambda < \lambda_0$. Although the critical value λ_0 depends on the mass value and gauge coupling value entering the Lagrangian, it is natural to consider small couplings to reach the Linde-Weinberg low bound. Physically, the Linde-Weinberg bound reflects the important property of the SSB – the existence of the parameter ranges allowing the total effective potential to be dominated by the positive radiation quantum effects instead of the negative classical part. Another example of degeneration of the SSB is the scalar model at finite temperatures. Studying the phase transition in the O(1) model on a lattice, we observed disappearance of the symmetry breaking at extremely small couplings [4]. Although there is no direct correspondence between the scalar O(N) model at zero temperature in the current paper and the mentioned examples of degeneration of the SSB, these analogies reflect the general idea that the SSB in theories with scalar fields can change its behavior at some extremal values of coupling λ.

5. Conclusions

We have investigated the SSB phenomenon in the scalar O(4) model in the four-dimensional space-time on a lattice and determined its dependence on the coupling values. We have shown that symmetry breaking takes place for $\lambda > \lambda_0 \approx 10^{-5}$. This is in the course of common belief that the SSB of continuous internal symmetries is a realistic mechanism for particle mass generation. It was experimentally grounded due to the recent discovery of the Higgs boson.

The main object introduced in the present paper is the effective Lagrangian for the radial field obtained by means of integration over angular continuous variables in the spherical coordinates in the internal space. The integration can be realized by the saddle-point method. The effective Lagrangian is convenient for further lattice investigations.

The open-source software package QCDGPU was used for MC simulations on GPU. We also propose the MC procedure for studying the dependence of the SSB phenomenon on the coupling constant λ values in the O(N) models. Within these facilities, we detected that the scalar condensate has to disappear at extremely small values of coupling, $\lambda < 10^{-5}$ for the O(4) model. This is in correspondence with the results obtained for the O(1) model [4], where the critical value is also $\lambda_0 \approx 10^{-5}$. For $N > 1$ this behavior means that the SSB of continuous symmetries also depends on the value of coupling.

We would like to stress here that the dependence of the SSB on the coupling value is a nonperturbative effect which cannot be expected beforehand. This also concerns changing the kind of the temperature phase transition. In the literature on this problem, most results have been obtained firstly in various perturbation schemes including resummations series of Feynman diagrams. It was observed that the dependence on λ disappears at all and perturbation theory in this parameter becomes not reliable at the critical temperature. So, other perturbation schemes (in particular, the expansion in $1/N$ for large N) have been used. In such investigations the dependence on the λ value was not considered. The coupling was usually taken to be of order $\lambda \approx 0.01 - 0.1$, and a second order phase transition has been detected. Such a behavior is in agreement with our analysis for O(1) model on a lattice [4] for these coupling values. The extremely small λ values were not considered at all.

In course of the results obtained, obvious problems for further studies are the λ-dependence of the SSB in the O(N) models for $d = 1$ - 3 and the temperature phase transition.

References

1. **Zinn-Justin J.** Quantum field theory and critical phenomena [Text] / J. Zinn-Justin // Int. Ser. Monogr. Phys. – 1996. – 92. – № 1. – P. 1008.
2. **Berges J.** Nonperturbative renormalization flow in quantum field theory and statistical physics [Text] / J. Berges, N. Tetradis and C. Wetterich // Phys. Rept. – 2002. – 363. – P. 223.
3. **Marky G.** Thermodynamics and phase transition of the O(N) model from the two-loop Phi-derivable approximation [Text] / G. Marky, U. Reinosa and Z. Szep // Phys. Rev. D. – 2013. – 87. – 105001.
4. **Bordag M.** The type of the phase transition and coupling values in $\lambda\varphi^4$ model [Text] / M. Bordag, V. Demchik, A. Gulov and V. Skalozub // Int. J. Mod. Phys. A. – 2012. – 27. – 1250116.
5. **Demchik V.** QCDGPU: Open-Source Package for Muti-GPU Monte Carlo Lattice Simulations [Text] / V. Demchik and N. Kolomoyets // Computer Science and Applications. – 2014. – V. 1. – N. 1. – P. 13–21.
6. **Demchik V.** Pseudorandom numbers generation for Monte Carlo simulations on GPU's: OpenCL approach [Text] / Numerical Computations with GPU's (ed. V. Kindratenko), Springer 2014, Ch. 12.
7. http://hgpu.org/.
8. **Linde A. D.** Dynamical Symmetry Restoration And Constraints On Masses And Coupling Constants in Gauge Theories [Text] / A.D. Linde // JETP Lett. – 1976. – 23. – P. 64.
9. **Weinberg S.** Mass of the Higgs Boson [Text] / S. Weinberg // Phys. Rev. Lett. – 1976. – 36. – P. 294.

Received 23.06.2015.