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M. P. Korkina, O. O. Iegurnov

Oles Honchar Dnipropetrovsk National University

MATCHING OF STEPHANI AND DE SITTER SOLUTIONS ON THE HYPERSURFACE OF CONSTANT TIME

The spherically symmetric solution for perfect fluid with homogeneous energy density and inhomogeneous pressure is considered. This solution is known as Stephani solution. It is reobtained by a mass-function method. Also, the meaning of arbitrary functions which are present in the solution is discussed. The matching of this solution and the de Sitter is done on a hypersurface of constant time. The matching is done with the Lichnerowicz – Darmois conditions. The coordinates of the de Sitter solution are taken in a general form as arbitrary functions that depend on Stephani's time and the radial coordinate. The matching is done both for the special cases of the flat, open, and closed Universe and for the general case, which does not concretize the type of curvature. The equality of energy densities and the abrupt change of pressure are observed on the matching hypersurface. Also, restrictions for arbitrary functions (coordinates of de Sitter solution) are found.

Keywords: perfect fluid, Stephani solution, de Sitter solution, Lichnerowicz - Darmois conditions.

Рассматривается сферически симметричное решение для идеальной жидкости с однородной плотностью энергии и неоднородным давлением. Это решение – известное решение Стефании. Это решение получается повторно с помощью метода массовой функции. Обсуждается смысл произвольных функций, которые присутствуют в решении. Сшивка этого решения с решением де Ситтера выполняется по гиперповерхности постоянного времени. Сшивка производится с помощью условий Лихнеровича-Дармуа. Координаты решения де Ситтера выбираются в общем виде, как произвольные функции от координаты времени и пространственной радиальной координаты решения Стефани. Сшивка выполняется для частных случаев плоской, закрытой, открытой Вселенной и для общего случая, в котором не конкретизируется кривизна. На гиперповерхности сшивки наблюдаются равенство плотностей энергий и скачок давления. Устанавливаются ограничения на произвольные функции – координаты решения де Ситтера.

Ключевые слова: идеальная жидкость, решение Стефани, решение де Ситтера, условия Лихнеровича – Дармуа.

Розглядається сферично симетричний розв'язок для ідеальної рідини з однорідною густиною енергії та неоднорідним тиском. Цей розв'язок - відомий розв'язок Стефані. Цей розв'язок встановлюється повторно за допомогою метода масової функції. Обговорюється значення довільних функцій, які присутні у розв'язку. Зшивка цього розв'язку з розв'язком де Сіттера здійснюється на гіперповерхні постійного часу. Зшивка здійснюється за допомогою умов Ліхнеровіча-Дармуа. Координати для розв'язку де Сіттера обираються у загальному вигляді, як довільні функції від координати часу та просторової радіальної координати розв'язку Стефані. Зшивка здійснюється для окремих випадків плоского, закритого і відкритого Всесвіту та для загального випадку, у якому не конкретизується кривизна. На гіперповерхні зшивки спостерігаються рівність густин енергій та стрибок тиску. Встановлюються обмеження на довільні функції – координати розв'язку де Сіттера.

Ключові слова: ідеальна рідина, розв'язок Стефані, розв'язок де Сіттера, умови Ліхнеровича – Дармуа.

Introduction

The most general class of non-static, perfect fluid solutions of Einstein's equations that are conformally flat is known as the "Stephani Universe" [1-5]. The spherically symmetric Stephany solution has been examined in numerous papers. A comprehensive review is presented in [5]. There are many papers devoted to applying this solution as star models, as generalization of the FLRW, and as a cosmological model [5-7]. In our opinion, this solution is attractive for the cosmological model for many reasons. Firstly, it is shear-free and inhomogeneous. The absence of a shear makes it simple for the cosmological purpose. The assumption of homogeneity is just a first approximation introduced to simplify Einstein's equations. So far this assumption has worked well, but future and modern observations can not be precise without taking into account inhomogeneity. And due to the fact, that modern and future observation data become more and more precise and that the smallest deviations from the standard model can be detected with high level of accuracy soon, makes inhomogeneous models actual. Secondly, Stephani solution has a general form in contrast to the FLRW solution where three solutions (flat, open, closed), non-transforming into each other, exist. Thirdly, the spatial curvature of this solution depends on time only via an arbitrary function, this fact is discussed in [3, 8].

The physical interpretation of the Stephani Universe is obscure. It is due to the many arbitrary functions and peculiar inhomogeneity – inhomogeneity is contained in pressure (depends on time and spatial coordinates), but density is homogeneous (depends on time only). It needs matching in order to determine some arbitrary functions. May be the main reason to use this solution in cosmological modeling is the fact that it is the generalization of the FLRW solution and, in our opinion, investigation of a more general solution is promising. The solution generalizes not only the FLRW but also the de Sitter solution [3]. In this connection, the idea to examine the Stephani solution on the de Sitter background looks reasonable.

In the first part of the artice the solution for perfect fluid with inhomogeneous pressure (the Stephani solution) is reobtained with the mass-function method [9-12]. Some properties of it are discussed. In particular, we discuss a sense of the arbitrary functions and transformation to the FLRW and to the de Sitter solutions.

In the second part, the matching of de Sitter and Stephani solutions on the hypersurface of constant time is done. The Lichnerowicz – Darmois conditions were used. Some consequences of the matching are discussed.

Mass-function method

The mass-function method essentially simplifies the appearance of the Einstein equations in contrast to the standard one; it makes them easier for work. The mass-function was introduced in [9] and discussed in [10-12]. As shown in [9], the mass-function is invariant and in our consideration may be determined as full energy limited by some hypersurface of constant time and coordinates. For a spherically symmetric metric:

$$dS^2 = e^{\nu(R,t)}dt^2 - e^{\lambda(R,t)}dR^2 - r^2(R,t)d\sigma^2, \tag{1}$$

where $d\sigma^2 = d\theta^2 - \sin^2(\theta)d\phi^2$, the mass-function $m(R,t)$ is:

$$m(R,t) = r(R,t)(1 + e^{\Phi} - e^{\Omega}), \tag{2}$$

$$e^{\Phi} = e^{-\nu}\dot{r}^2(R,t), \quad e^{\Omega} = e^{-\lambda}r'^2(R,t), \tag{3}$$

where $\dot{r}(R,t) = \dfrac{\partial r(R,t)}{\partial t}$, $r'(R,t) = \dfrac{\partial r(R,t)}{\partial R}$.

Einstein's equations with the mass-function have the form:

$$\begin{cases} m' = r^2 r' T_0^0, \\ \dot{m} = r^2 \dot{r} T_1^1, \\ 2\dot{m}' = \dot{m}\Phi' + m'\dot{\Omega} + 4r\dot{r}r' T_2^2, \\ 2\dot{r}' = \dot{r}\Phi' + r'\dot{\Omega}. \end{cases} \tag{4}$$

Obtaining the Stephani solution

This solution was first found by Stephani [13] as a special example of a space-time embeddable in a flat five-dimensional space, and later reobtained by Krasinski [2]. We reobtained this solution with mass-function method.

The Stephani solution is an isotropic solution for perfect fluid with homogeneous density $\rho = \rho(t)$ and inhomogeneous pressure $p = p(R,t)$ (in spherically symmetric consideration). The stress-energy tensor for such perfect fluid is: $T_0^0 = \rho(t)$, $T_1^1 = T_2^2 = T_3^3 = -p(R,t)$. The Einstein field equations become

$$\begin{cases} m' = r^2 r' \rho, \\ \dot{m} = -r^2 \dot{r} p, \\ 2\dot{m}' = \dot{m}\Phi' + m'\dot{\Omega} - 4r\dot{r}r' p, \\ 2\dot{r}' = \dot{r}\Phi' + r'\dot{\Omega}. \end{cases} \tag{5}$$

Expressing the mass-function from the first equation of the set and substituting it into the third one gives

$$\frac{\dot{\rho}}{\rho}\left(\frac{1}{3}r\Phi' + \frac{4}{3}r' - 2r'\right) = 0, \tag{6}$$

from this equation the expression for Φ is obtained,

$$\Phi = \ln r^2 \psi^2, \tag{7}$$

where $\psi = \psi(t)$ is an arbitrary function of integration.

The fourth equation of the set (5) gives us the expression for Ω:

$$\Omega = \ln \frac{r'^2}{r^2 k'^2}, \tag{8}$$

where $k = k(R)$ - arbitrary function (prime is used for convenience).

From Eqs. (1) and (3) the metric is obtained,

$$dS^2 = \frac{\dot{r}^2}{r^2 \psi^2} dt^2 - r^2 (k'^2 dR^2 + d\sigma^2). \tag{9}$$

With the expressions (7) and (8) the mass-function is obtained:

$$m = r(1 + r^2\psi^2 - \frac{r'^2}{r^2 k'^2}). \tag{10}$$

The expression (10) with the first equation of (1.5) gives us

$$\rho \frac{r^3}{3} = r + r^3\psi^2 - \frac{r'^2}{rk'^2}. \tag{11}$$

It is integrated in elementary functions providing the expression for $r = r(R,t)$

$$r(R,t) = 2(e^{k(R)+\eta(t)} - \zeta(t)e^{-k(R)-\eta(t)})^{-1}, \tag{12}$$

$$\zeta(t) = \psi^2(t) - \frac{1}{3}\rho(t), \tag{13}$$

and $\eta(t)$ is an arbitrary function of integration.
Depending on the sign of $\zeta(t)$, the expression (12) gives

$$\begin{aligned} r^{-1} &= \sqrt{\zeta}\cdot\sinh(k+\alpha),\ \zeta > 0, \\ r^{-1} &= \sqrt{|\zeta|}\cdot\cosh(k+\alpha),\ \zeta < 0, \\ r^{-1} &= e^{k+\alpha},\ \zeta = 0, \end{aligned} \tag{14}$$

where $e^{\eta} = \sqrt{|\zeta|}\cdot e^{\alpha}$.

In contrast to the FLRW solution where there are three non-transforming into each other solutions (flat, closed, open), there is the general solution here with flat, open, closed solutions as special cases. The existence of this solution shows that the distinction between the closed and open Universe is not required by Einstein's theory of gravitation as such, but is due to the very strong symmetry assumptions that are set into the models just from the beginning. From (5) it is also possible to obtain the equation that links density and pressure:

$$p(R,t) = -\rho(t) - \frac{\dot{\rho}}{3}\frac{r(R,t)}{\dot{r}(R,t)}. \tag{15}$$

The FLRW and de Sitter solutions as special cases of the Stephani solution

The Stephani solution, as mentioned above, is a generalization of the FLRW solution and the de Sitter one. When $\zeta(t) = 0$, the Stephani solution is transformed to the flat FLRW solution. If $\zeta(t) \neq 0$, the transformation occurs in such a way

$$\begin{aligned} &\dot{\eta}(t) = 0, \psi(t) = \frac{\dot{a}(t)}{a(t)}, \\ &\zeta(t) = -\frac{1}{a^2(t)} : k(R) = \ln\cot\frac{R}{2}, \\ &\zeta(t) = \frac{1}{a^2(t)} : k(R) = \ln\coth\frac{R}{2}. \end{aligned} \tag{16}$$

The density in the de Sitter solution is $\rho = \frac{1}{a_\lambda{}^2} = const$. It can be obtained from Eqs. (13, 16):

$$\begin{aligned} &\zeta(t) = 0 : a(t) = e^{\frac{t}{a_\lambda}}, \\ &\zeta(t) = -\frac{1}{a^2(t)} : a(t) = a_\lambda \cosh\frac{t}{a_\lambda}, \\ &\zeta(t) = \frac{1}{a^2(t)} : a(t) = a_\lambda \sinh\frac{t}{a_\lambda}. \end{aligned} \tag{17}$$

Arbitrary functions and their meaning

The Stephani solution contains four arbitrary functions: $k(R), \zeta(t), \psi(t), \eta(t)$. Also in our consideration, we do not set the equation of state, i.e. $\rho(t)$ is undefined. The determination of arbitrary functions may be proving to be elusive. It is true, but our analysis of the solution (9) shows that it is possible to understand their meaning.

The function $k(R)$ may be chosen arbitrary because it leads to a transition to another coordinate system, only. The part $\left(\frac{dk}{dR}\right)^2 dR^2 = dk^2$ in the expression (9) is just a transformation from R to k. The coordinate transformation may be chosen in such a way that the spatial part of the solution (9) is conformal to one of three homogeneous and isotropic spaces,

$$\begin{aligned} dk^2 + d\sigma^2 &= \frac{1}{\sinh^2(R_1)}\left(dR_1^2 + \sinh^2\left(R_1\right)d\sigma^2\right) = \\ &= \frac{1}{\sin^2(R_2)}\left(dR_2^2 + \sin^2\left(R_2\right)d\sigma^2\right) = \frac{1}{R_3^2}\left(dR_3^2 + R_3^2 d\sigma^2\right). \end{aligned} \tag{18}$$

Chosen $\eta(t)$ is also referring to the coordinate transformation. Thus Eq. (9) takes the form

$$dS^2 = \frac{r_\eta^2}{r^2\psi^2}d\eta^2 - r^2(k'^2 dR^2 + d\sigma^2) \tag{19}$$

where $r_\eta = \frac{\partial r}{\partial \eta}$.

The analysis of invariants of the spatial curvature tensor of the metric (9) shows that the invariants depend on the arbitrary function $\zeta(t)$, only. The scalar curvature tensor and the Kretschmann scalar, for example, are

$$\begin{aligned} R &= 6\zeta(t), \\ R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} &= 12\zeta^2(t). \end{aligned} \tag{20}$$

Thereby spatial curvature depends on $\zeta(t)$ only. The type of space (flat, open, closed) is determined by the sign of $\zeta(t)$. So it is possible to assume that $\zeta(t)$ completely determines spatial curvature. This fact is also discussed in [3, 8].

Thus, $\psi(t)$ obtains the meaning of critical energy density. When spatial curvature is zero, then $\zeta(t) = 0$ and from Eq. (13) we have

$$\psi^2(t) = \frac{1}{3}\rho_c(t). \tag{21}$$

Lichnerowicz - Darmois conditions

The Lichnerowicz-Darmois matching conditions [14] are two metrics

$$\begin{aligned} dS_1^2 &= g_{\mu'\nu'}dx^{\mu'}dx^{\nu'}, \\ dS_2^2 &= g_{\mu\nu}dx^{\mu}dx^{\nu}, \end{aligned} \tag{22}$$

and are said to match across some hypersurface if first and second fundamental forms of this hypersurface are identical for the two metrics.

The first fundamental form is

$$dl_1^{\,2} = a_{ik} du^i du^k \quad (i,k = 1,2,3), \tag{23}$$

where $a_{ik} = g_{\mu\nu}\xi_i^\mu \xi_k^\nu$, $\xi_i^\mu = \dfrac{\partial x^\mu}{\partial x^i}$.

The second fundamental form is

$$dl_2^{\,2} = b_{ik} du^i du^k, \tag{24}$$

where $b_{ik} = \nu_{\mu;\nu}\xi_i^\mu \xi_k^\nu$, ξ_i^μ are tangent vectors to the hypersurface and ν^μ are normal vectors.

Matching Stephani and de Sitter solutions in general case

The matching has been done on the hypersurface of Stephani's constant time. Time and spatial coordinate of the de Sitter solution were chosen as arbitrary functions of the Stephani time and the radial coordinate. Both metrics have been taken in the general form without more precise definition of their curvature.

The Stephani metric, as mentioned above, is

$$dS_{st}^{\;2} = \frac{\dot{r}^2}{r^2\psi^2} d\tau^2 - r^2(dk^2 + d\sigma^2), \tag{25}$$

where $r = r(k,\tau)$, $\psi = \psi(\tau)$, $\dot{r} = \dfrac{\partial r}{\partial \tau}$.

The matching is performed on the hypersurface $\tau = const$. ξ_i^μ on this hypersurface is

$$\xi_i^\mu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

Normal vectors are found from the equation

$$\begin{cases} \nu_\mu \nu^\mu = 1, \\ \nu_\mu \xi_i^\mu = 0. \end{cases} \tag{27}$$

The normal vector has only non-zero component

$$\nu_0 = \frac{\dot{r}}{r\psi}. \tag{28}$$

First and second fundamental forms for the Stephani solution are

$$dl_{st1}^2 = r^2 dk^2 + r^2 d\sigma^2, \tag{29}$$

$$dl_{st2}^2 = r^2\psi dk^2 + r^2\psi d\sigma^2. \tag{30}$$

The de Sitter metric is

$$dS_{ds}^{\;2} = \left(1 - \frac{r_s^{\,2}}{a_\lambda^{\,2}}\right) dt^2 - \frac{1}{1 - \dfrac{r_s^{\,2}}{a_\lambda^{\,2}}} dr_s^{\,2} - r_s^{\,2} d\sigma^2, \tag{31}$$

$r_s = r_s(k,\tau),\ \ t = t(k,\tau),\ \ a_\lambda = \sqrt{\dfrac{3}{\Lambda}},$ and Λ is cosmological constant. We do the same manipulations with the de Sitter metric

$$\xi_i^\mu = \begin{pmatrix} 0 & t' & 0 & 0 \\ 0 & r_s' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{32}$$

According to Eq. (27) we have

$$v_0 = \frac{r_s'}{\sqrt{A^{-1}r_s'^2 - At'^2}}, \tag{33}$$

$$v_1 = \frac{t'}{\sqrt{A^{-1}r_s'^2 - At'^2}}. \tag{34}$$

where primes denote derivatives for k , $A = 1 - \dfrac{r^2{}_s}{a_\lambda{}^2}$.

The first and the second fundamental forms for the de Sitter solution are

$$dl_{ds1}{}^2 = (A^{-1}r_s'^2 - At'^2)dk^2 + r_s{}^2 d\sigma^2, \tag{35}$$

$$dl_{ds2}{}^2 = \frac{r_s t'(t'^2 A - 3r_s'^2 A^{-1})}{a_\lambda{}^2\sqrt{A^{-1}r_s'^2 - At'^2}}dk^2 + \frac{r_s At'}{\sqrt{A^{-1}r_s'^2 - At'^2}}d\sigma^2. \tag{36}$$

From the equality of the first and second fundamental forms the following equations cane be obtained

$$r^2 = r_s{}^2, \tag{37}$$

$$r^2 = A^{-1}r_s'^2 - At'^2, \tag{38}$$

$$r^2\psi = \frac{r_s t'(t'^2 A - 3r_s'^2 A^{-1})}{\sqrt{A^{-1}r_s'^2 - At'^2}}, \tag{39}$$

$$r^2\psi = \frac{r_s At'}{\sqrt{A^{-1}r_s'^2 - At'^2}}. \tag{40}$$

From these matching conditions the equality of energy densities on the hypersurface $\tau = const$ follow. From Eqs. (39) and (40) we have

$$r_s At' = \frac{r_s}{a_\lambda{}^2}t'(t'^2 A - 3r_s'^2 A^{-1}), \tag{41}$$

and with Eq. (38)

$$A = \frac{1}{a_\lambda{}^2}(-r_s{}^2 - 2r_s'^2 A^{-1}),$$

$$r_s'^2 = \frac{r_s{}^2 - a_\lambda{}^2}{2},$$

with Eqs. (37) and (32) we obtain

$$\rho_{st} = \frac{1}{a_\lambda{}^2}. \tag{42}$$

But this is de Sitter's energy density. So, on the hypersurface $\tau = const$ the equality of energy densities holds

$$\rho_{st} = \rho_{ds}. \tag{43}$$

Matching Stephani and de Sitter solutions in the flat case

The de Sitter metric for the flat case has the form

$$dS_{ds}{}^2 = dt^2 - a_\lambda{}^2 e^{\frac{2t}{a_\lambda}} (dr_s^2 + r_s^2 d\sigma^2). \tag{44}$$

After transforming its spatial part to the convenient form, it takes the form

$$dS_{ds}{}^2 = dt^2 - a_\lambda{}^2 e^{\frac{2t}{a_\lambda}+2X} (dX^2 + d\sigma^2). \tag{45}$$

The Stephani metric is

$$dS_{st}{}^2 = \frac{1}{\psi^2} d\tau^2 - 4e^{-2(R-\tau)} (dR^2 + d\sigma^2). \tag{46}$$

As mentioned, $t = t(\tau, R),\ X = X(\tau, R)$. Below dots and primes mean derivatives with respect to the time and the radial coordinate, respectively. For the Stephani solution the normal vector has one non-zero component on the matching hypersurface $\tau = const$:

$$\nu_0 = \frac{1}{\psi}. \tag{47}$$

For the de Sitter solution the normal vector has two non-zero components

$$\nu_0 = \frac{X' a_\lambda e^{\frac{t}{a_\lambda}+X}}{\sqrt{X'^2 a_\lambda{}^2 e^{2(\frac{t}{a_\lambda}+X)} - t'^2}}, \tag{48}$$

$$\nu_1 = -\frac{t' a_\lambda e^{\frac{t}{a_\lambda}+X}}{\sqrt{X'^2 a_\lambda{}^2 e^{2(\frac{t}{a_\lambda}+X)} - t'^2}}, \tag{49}$$

From the equality of the first fundamental forms two matching conditions can be obtained

$$4e^{-2(R+\tau)} = X'^2 a_\lambda{}^2 e^{2(\frac{t}{a_\lambda}+X)} - t'^2, \tag{50}$$

$$4e^{-2(R+\tau)} = a_\lambda{}^2 e^{2(\frac{t}{a_\lambda}+X)} (X'^2 - 1). \tag{51}$$

Right-hand sides of these equations are equal; from this equality we obtain

$$t'^2 = a_\lambda{}^2 e^{2(\frac{t}{a_\lambda}+X)} (X'^2 - 1). \tag{52}$$

From the equality of the second fundamental forms such two matching conditions follow:

$$4\psi e^{-2(R+\tau)} = -\frac{a_\lambda e^{\frac{t}{a_\lambda}+X}}{\sqrt{X'^2 a_\lambda{}^2 e^{2(\frac{t}{a_\lambda}+X)} - t'^2}}\left(a_\lambda X' e^{2(\frac{t}{a_\lambda}+X)} + t'\right), \tag{53}$$

$$4\psi e^{-2(R+\tau)} = -\frac{e^{\frac{t}{a_\lambda}+X}}{\sqrt{X'^2 a_\lambda{}^2 e^{2(\frac{t}{a_\lambda}+X)} - t'^2}}(-\dot{X}' a_\lambda t'^2 + X' a_\lambda \dot{t}' t' + X'^2 a_\lambda t'' - \tag{54}$$

$$-a_\lambda t' X' X'' + X'^3 a_\lambda{}^2 e^{2(\frac{t}{a_\lambda}+X)} - X'^2 a_\lambda t' - 2t'^2 X'),$$

Equality of right-hand sides of Eqs. (53) and (54) together with Eq. (52) gives us

$$(X'^2 - 1)(1 - t'X'\dot{X} - t'X'^3 + X') + e^{\frac{t}{a_\lambda}+X}(X'^2 - 1)^{\frac{3}{2}}(X' - X'^2 - \dot{t}X') + \tag{55}$$
$$+(-\dot{X}'t' - X'X'') = 0$$

Two possibilities exist in this case
the first is

$$X'^2 = 1, \tag{56}$$

and the second one reads

$$\begin{cases} 1 - t'X'\dot{X} - t'X'^3 + X' = 0, \\ X' - X'^2 - \dot{t}X' = 0, \\ \dot{X}'t' + X'X'' = 0. \end{cases} \tag{57}$$

But Eq. (57) is an incompatible system. So, we conclude that $X = X(R) = R + const$, and from Eq. (52) $t = t(\tau)$.

Matching Stephani and de Sitter solutions in the open case

We take the de Sitter and Stephani metrics in the open case in the form

$$dS_{ds}{}^2 = dt^2 - \frac{a_\lambda{}^2 \sinh^2\left(\frac{t}{a_\lambda}\right)}{\sinh^2 X}(dX^2 + d\sigma^2), \tag{58}$$

$$dS_{st}{}^2 = \frac{\left(\frac{1}{2}\frac{\dot{\zeta}}{\zeta} + \coth(R+\tau)\right)^2}{\psi^2} d\tau^2 + \frac{1}{\zeta \sinh^2(R+\tau)}(dR^2 + d\sigma^2), \tag{59}$$

where $t = t(\tau, R),\ X = X(\tau, R)$.

For the Stephani solution normal vector has the only non-zero component on the hypersurface $\tau = const$:

$$\nu_0 = \frac{\frac{1}{2}\frac{\dot{\zeta}}{\zeta} + \coth(R+\tau)}{\psi}. \tag{60}$$

The non-zero components of the normal vector for the de Sitter solution are

$$\nu_0 = \frac{X' a_\lambda \sinh \dfrac{t}{a_\lambda}}{\sqrt{X'^2 a_\lambda{}^2 \sinh^2 \dfrac{t}{a_\lambda} - t'^2 \sinh^2 X}}, \tag{61}$$

$$\nu_1 = -\frac{t' a_\lambda \sinh \dfrac{t}{a_\lambda}}{\sqrt{X'^2 a_\lambda{}^2 \sinh^2 \dfrac{t}{a_\lambda} - t'^2 \sinh^2 X}}. \tag{62}$$

From the equality of the first fundamental forms, the conditions follow:

$$\frac{1}{\zeta \sinh^2 (R+\tau)} = X'^2 a_\lambda{}^2 \frac{\sinh^2 \dfrac{t}{a_\lambda}}{\sinh^2 X} - t'^2, \tag{63}$$

$$\frac{1}{\zeta \sinh^2 (R+\tau)} = a_\lambda{}^2 \frac{\sinh^2 \dfrac{t}{a_\lambda}}{\sinh^2 X}. \tag{64}$$

From the equality of the right-hand sides of Eqs. (63) and (64) we have

$$t'^2 = a_\lambda{}^2 \frac{\sinh^2 \dfrac{t}{a_\lambda}}{\sinh^2 X} (X'^2 - 1). \tag{65}$$

From the equality of the second fundamental forms, the conditions follow:

$$\frac{\psi}{\zeta \sinh^2 (R+\tau)} = \frac{a_\lambda \sinh \dfrac{t}{a_\lambda}}{\sinh^2 X \sqrt{X'^2 a_\lambda{}^2 \sinh^2 \dfrac{t}{a_\lambda} - t'^2 \sinh^2 X}} \times$$
$$\times (t' \cosh X \sinh X - aX' \sinh \frac{t}{a_\lambda} \cosh \frac{t}{a_\lambda}), \tag{66}$$

$$\frac{\psi}{\zeta \sinh^2(R+\tau)} = \frac{1}{\sqrt{X'^2 a_\lambda{}^2 \sinh^2 \frac{t}{a_\lambda} - t'^2 \sinh^2 X}} (a_\lambda \dot{X} t'^2 \sinh \frac{t}{a_\lambda} -$$

$$-a_\lambda \dot{t}' X' t' \sinh \frac{t}{a_\lambda} + X' t'^2 \cosh \frac{t}{a_\lambda} - \frac{a_\lambda t' X'^2 \cosh X \sinh \frac{t}{a_\lambda}}{\sinh X} - \tag{67}$$

$$-a_\lambda X'^2 t'' \sinh \frac{t}{a_\lambda} + a_\lambda X'' t' X' \sinh \frac{t}{a_\lambda} - \frac{a_\lambda{}^2 X'^3 \cosh \frac{t}{a_\lambda} \sinh^2 \frac{t}{a_\lambda}}{\sinh^2 X} +$$

$$+ X' t'^2 \cosh \frac{t}{a_\lambda}).$$

The equality of the right-hand sides of Eqs. (66) and (67) gives

$$\cosh \frac{t}{a_\lambda} (X'^2 - 1)^{\frac{3}{2}} (X' - X'^2 - \dot{t} X') + \sinh X(-X'' X' - \dot{X} t') +$$
$$+ \cosh X (X'^2 - 1)(-1 - X'^2) + \cosh X (X'^2 - 1)^{\frac{3}{2}} (X'^3 + \dot{X} X' t') = 0. \tag{68}$$

Two possibilities exist for satisfying this equation:

1) $$X'^2 = 1, \tag{69}$$

or

2) $$\begin{cases} X' - X'^2 - \dot{t} X' = 0, \\ X'' X' + \dot{X} t' = 0, \\ 1 + X'^2 = 0, \\ X'^3 + \dot{X} X' t' = 0. \end{cases} \tag{70}$$

The last set of equations is an incompatible system, so, we conclude from Eqs. (69), (65) that $X = X(R) = R + const$ and $t = t(\tau)$.

Matching Stephani and de Sitter solutions in the closed case

Now, we take the de Sitter and Stephani metrics in a more convenient for our purpose form

$$dS_{ds}{}^2 = dt^2 - a_\lambda{}^2 \frac{\cosh^2 \frac{t}{a_\lambda{}^2}}{\cosh^2 X} (dX^2 + d\sigma^2), \tag{71}$$

$$dS_{st}^2 = \frac{\left(\frac{1}{2} \frac{\dot{\zeta}}{\zeta} + tanh(R+r) \right)^2}{\psi^2} d\tau^2 - \frac{1}{\zeta\, cosh^2(R+h)} (dR^2 + d\sigma^2). \tag{72}$$

We match the functions $t=t(\tau,R), X=X(\tau,R)$ on the hypersurface $\tau = const$. Dots and primes denote derivatives with respect to time and radial coordinate, respectively.

The non-zero component for the normal vector in the Stephani case on the matching hypersurface is

$$\nu_0 = \frac{\frac{1}{2}\frac{\dot{\zeta}}{\zeta} + \tanh(R+\tau)}{\psi}. \tag{73}$$

And non-zero components for the de Sitter case are:

$$\nu_0 = \frac{X' a_\lambda \cosh\frac{t}{a_\lambda}}{\sqrt{X'^2 a_\lambda{}^2 \cosh^2\frac{t}{a_\lambda} - t'^2 \cosh^2 X}}, \tag{74}$$

$$\nu_1 = -\frac{t' a_\lambda \cosh\frac{t}{a_\lambda}}{\sqrt{X'^2 a_\lambda{}^2 \cosh^2\frac{t}{a_\lambda} - t'^2 \cosh^2 X}}. \tag{75}$$

From the equality of the first fundamental forms, the conditions follow

$$\frac{1}{\zeta\cosh^2(R+\tau)} = t'^2 - X'^2 a_\lambda{}^2 \frac{\cosh^2\frac{t}{a_\lambda}}{\cosh^2 X}, \tag{76}$$

$$\frac{1}{\zeta\cosh^2(R+\tau)} = -\frac{a_\lambda{}^2\cosh^2\frac{t}{a_\lambda}}{\cosh^2 X}, \tag{77}$$

From the equality of the right-hand sides of Eqs. (76) and (77) we get

$$t'^2 = \frac{a_\lambda{}^2\cosh^2\frac{t}{a_\lambda}}{\cosh^2 X}(X'^2 - 1). \tag{78}$$

From the equality of the second fundamental forms, the conditions follow:

$$\frac{\psi}{\zeta\cosh^2(R+\tau)} = \frac{a_\lambda \cosh\dfrac{t}{a_\lambda}}{\cosh^2 X\sqrt{X'^2 a_\lambda{}^2\cosh^2\dfrac{t}{a_\lambda} - t'^2\cosh^2 X}} \times$$
$$\times(a_\lambda X'\cosh\frac{t}{a_\lambda}\sinh\frac{t}{a_\lambda} - t'\sinh X\cosh X), \tag{79}$$

$$\frac{\psi}{\zeta\cosh^2(R+\tau)} = \frac{1}{\sqrt{X'^2 a_\lambda{}^2\cosh^2\dfrac{t}{a_\lambda} - t'^2\cosh^2 X}}(a_\lambda t'\cosh\frac{t}{a_\lambda}\times$$
$$\times(\dot{t}'X' - t'\dot{X}') + a_\lambda X'\cosh\frac{t}{a_\lambda}(t''X' - X''t') + X'\sinh\frac{t}{a_\lambda}\frac{1}{\cosh^2 X}\times \tag{80}$$
$$\times(X'^2 a_\lambda{}^2\cosh^2\frac{t}{a_\lambda} - t'^2\cosh^2 X) + t'X'^2 a_\lambda\cosh\frac{t}{a_\lambda}\tanh X -$$
$$-X't'^2\sinh\frac{t}{a_\lambda}).$$

The equality of the right-hand sides of Eqs. (79) and (80) gives

$$\cosh X(t'\dot{X}' + X''X') + \sinh X(X'^2 - 1)(1 + X'^2 - X'^3 - X'\dot{X}t') +$$
$$+X'\sinh\frac{t}{a_\lambda}(X'^2 - 1)^{\frac{3}{2}}(\dot{t} + X' - 1) = 0. \tag{81}$$

Two possibilities exist to satisfy this equation:
1)

$$X'^2 = 1, \tag{82}$$

or
2)

$$\begin{cases} t'\dot{X}' + X''X' = 0, \\ 1 + X'^2 - X'^3 - X'\dot{X}t' = 0, \\ \dot{t} + X' - 1 = 0. \end{cases} \tag{83}$$

This set of equations is incompatible, so, we conclude from Eqs. (82) and (78) that $X = X(R) = R + const$ and $t = t(\tau)$.

Conclusions

Matching conditions for the Stephani and the de Sitter solutions on hypersurface $\tau = const$ in the spherically symmetric case have been obtained (τ is a time coordinate of the Stephani solution). The coordinates of the de Sitter solution were taken in the general form as arbitrary functions depending on the Stephani's time and radial coordinate. Matching was done both for special cases (flat, open, closed) and for the general case that does not concretize the type of curvature. From the matching conditions the equality of densities on the matching hypersurface has been obtained. From Eq. (15) we see that there is an abrupt change of pressure. Also it was obtained that de Sitter radial coordinate is different from the Stephani one on some shift and de Sitter time is an arbitrary function depending on Stephani's time.

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